UNIVERSITY
OF MICHIGAN
DEC 8 1950
MATHEMATICS
LIBRARY

D
c; X

PROCEEDINGS

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

GUSTAV A. HEDLUND NATHAN JACOBSON

A. C. SCHAEFFER

WITH THE ASSISTANCE OF

C. B. ALLENDOERFER R. H. BING A. H. CLIFFORD
M. M. DAY J. B. DIAZ F. G. DRESSEL
G. P. HOCHSCHILD I. M. SHEFFER MAX SHIFFMAN
G. W. WHITEHEAD

VOLUME 1, NUMBER 5
OCTOBER, 1950

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK

Journals Published by the

American Mathematical Society

Bulletin of the American Mathematical Society

This journal is the official organ of the Society. It reports official acts of the Society and the details of its meetings, with abstracts of all research papers presented. It contains some of the officially invited addresses presented before the Society. It contains also announcements and notes on matters of current interest, and reviews of advanced mathematical books.

One volume of six numbers is planned for 1950. The subscription price for the BULLETIN and the PROCEEDINGS (see below) is now $11.00, and the two journals are provided together to subscribers, and cannot be obtained separately. For 1951 subscriptions may be placed for the BULLETIN separately and the subscription price will be $7.00. It is planned to publish one volume of six issues in 1951. Sets of volumes of the BULLETIN are available as follows: vols. 1–36 (1894–1930), $7.00 each; vols. 37–53 (1931–1947), $9.00 each; vols. 54–55 (1948–1949), $11.00 each.

Proceedings of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics, and is devoted principally to the publication of original papers of moderate length. It contains the material formerly published in the even-numbered issues of the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY. Volume 1 appears in 1950.

One volume of six numbers is planned for 1950. The PROCEEDINGS is supplied with the BULLETIN to all subscribers. See the announcement concerning the BULLETIN above. For 1951 one volume of six issues is planned and subscriptions may be placed for the PROCEEDINGS separately. The price will be $11.00.

Transactions of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics, and includes in general longer memoirs than the PROCEEDINGS.

Normally, two volumes of three numbers each are published annually. The price per volume is $6.00. Volumes 1–36 (1900–1934) consist of four numbers each. Volumes 66–67 (1949) consist of two numbers each. Sets of volumes are available as follows: vols. 1–21 (1900–1920), $7.00 each; vols. 22–32 (1921–1930), $8.00 each; vols. 33–36 (1931–1934), $9.00 each; vols. 37–68 (1935–1950), $6.00 each.

Mathematical Reviews

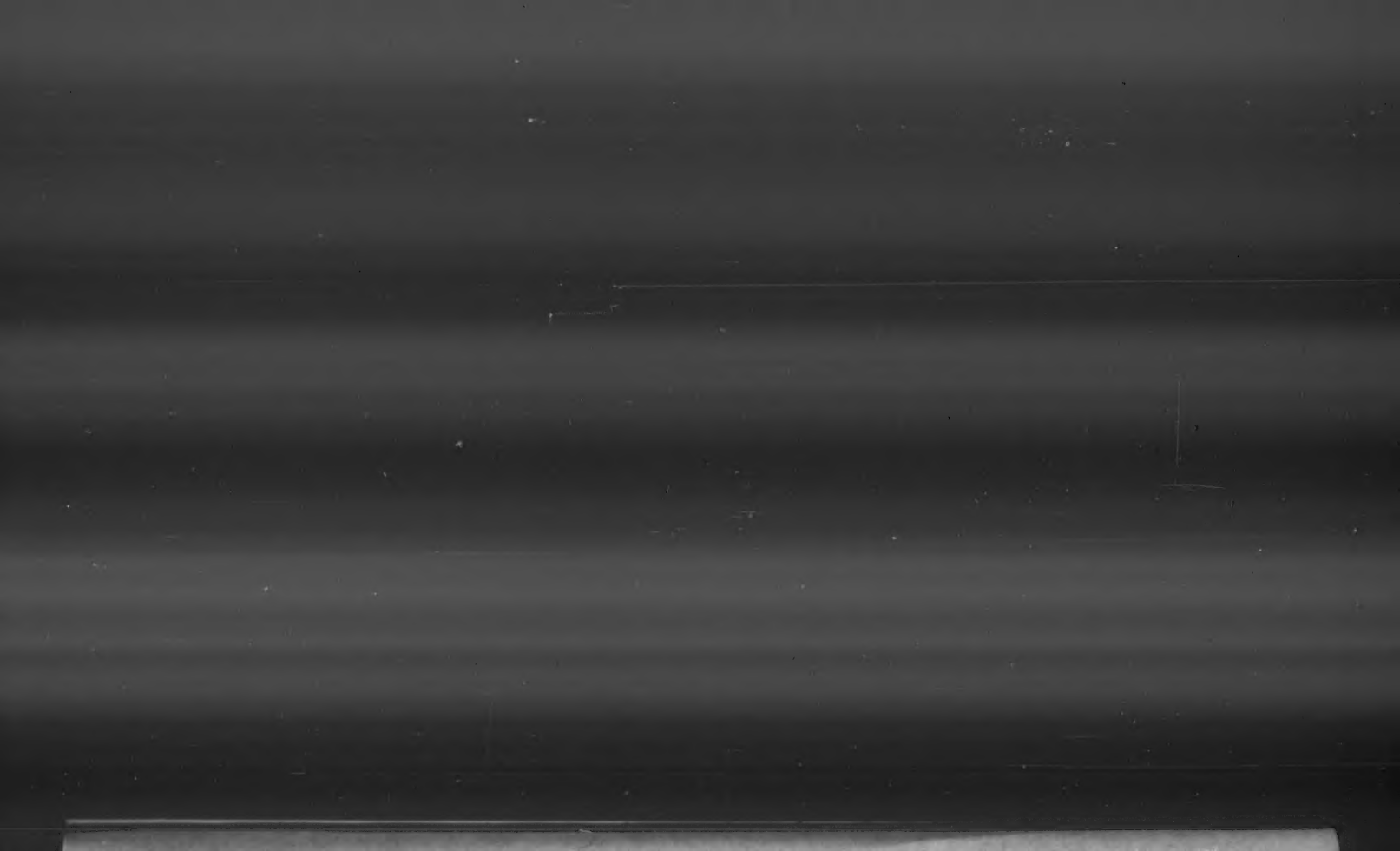
This journal contains abstracts and reviews of the current mathematical literature of the world. It is sponsored by thirteen mathematical organizations, located both in the United States and abroad.

The publication of MATHEMATICAL REVIEWS was begun in 1940. It appears monthly, excepting July, and the price per annual volume is $20.00. Orders for complete volumes only are accepted. Volumes 1–10 (1940–1949) are available at $20.00 per volume.

The PROCEEDINGS OF THE AMERICAN MATHEMATICAL SOCIETY is published bimonthly, in February, April, June, August, October, and December. Price per annual volume, with the BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, $11.00. Subscriptions, orders for back numbers, and inquiries in regard to nondelivery of current numbers should be addressed to the American Mathematical Society, 450 Ahnaip Street, Menasha, Wis., or Low Memorial Library, 531 West 116th Street, New York 27, N.Y.

Entered as second-class matter at the post office at Menasha, Wisconsin. Authorization is granted under the authority of the act of August 24, 1912, as amended by the act of August 4, 1947 (sec. 34.21, P. L. & R.).

Copyright, American Mathematical Society, 1950



A BASIS FOR FREE LIE RINGS AND HIGHER COMMUTATORS IN FREE GROUPS

MARSHALL HALL, JR.

1. **Introduction.** It has been shown by E. Witt[1] [4] that there is an isomorphism between free Lie rings[2] and the higher commutator groups of free groups. In terms of this isomorphism, the problem of finding a basis is the same in both instances. It is shown here that the commutators which arise in Philip Hall's collecting process [2] will serve as a basis. These are called standard commutators here, and the corresponding terms in the Lie ring are called standard monomials.

Standard monomials in a free Lie ring are defined in §2. In §3 a canonical process is given for reducing an arbitrary element of a Lie ring to a standard form, this being a linear combination of standard monomials. It is then proved that the standard monomials form a basis for the Lie ring. §4 restates the results of §3 in the appropriate form for higher commutators in free groups.

2. **The standard form for Lie products.** We shall write the product of x and y in a Lie ring as $[x, y]$. The laws satisfied by the Lie product are

$$[x, x] = 0, \tag{2.1.1}$$

$$[x, y] + [y, x] = 0, \tag{2.1.2}$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0. \tag{2.1.3}$$

For simpler notation we write $[x, y, z]$ for $[[x, y], z]$ and more generally $[u_1, \cdots, u_n]$ for $[[u_1, \cdots, u_{n-1}], u_n]$. We also write $[x, y; z, w]$ for $[[x, y], [z, w]]$. This is in accord with the notation used by Philip Hall [2] for higher commutators in groups.

Given an associative ring with product xy of x and y, we may put

$$[x, y] = xy - yx \tag{2.2}$$

to define a Lie ring on the same elements. It has been shown conversely by Garrett Birkhoff [1] and by E. Witt [4] that every Lie

Presented to the Society, November 26, 1949; received by the editors August 16, 1949.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

[2] In a strict sense the "coefficients" in a free Lie ring are the integers whose meaning is derived from $a+a+\cdots+a=na$ and $a+(-a)=0$. The treatment here is also valid for coefficients from any field or even from various commutative rings.

ring has a faithful representation by an associative ring using the rule (2.2). In particular [1, Theorem 3], the free Lie ring generated by $x_1, \cdots, x_q$ is faithfully represented by the free associative ring generated by $x_1, \cdots, x_q$. In the free associative ring a monomial is determined by the order of its factors, since all possible ways of inserting parentheses yield the same element. In this sense distinct monomials are distinct elements, and in fact form a basis for the ring. In the free Lie ring, every element may be written as a linear combination of the "left normed" elements $[u_1, \cdots, u_n]$ [cf. 3] but these are not independent. Thus $[x, y, x, y] = -[x, y; x, y] - [y; x, y; x] = [x, y, y, x]$.

Let L be the free Lie ring generated by $x_1, \cdots, x_q$. Then $x_1, \cdots, x_q$ are the monomials of degree one. If u is a monomial of degree r and v is a monomial of degree s, then $[u, v]$ is a monomial of degree $r+s$. Any linear combination of monomials of degree n will be said to be homogeneous of degree n. Every element of L may be written as a linear combination of monomials. If we put $x = u+v$, then (2.1.1) follows from $[u, u]=0$, $[v, u]+[u, v]=0$, and $[v, v]=0$. Also (2.1.2) and (2.1.3) follow from the corresponding relations with x replaced by u and v. Repeating this argument as often as necessary, we see that every relation in L is a consequence of the distributive laws and the rules (2.1) for monomials. In particular if

$$W = W_1 + \cdots + W_n = 0, \tag{2.3}$$

with each W_i homogeneous of degree i, then

$$W_1 = \cdots = W_n = 0. \tag{2.4}$$

There are only a finite number of monomials of given degree n. Thus we shall have a basis for L if we can exhibit for each n a number of monomials $U_{n1}, \cdots, U_{nr}$ of degree n such that:

(1) Every homogeneous expression of degree n may be written as a linear combination of $U_{n1}, \cdots, U_{nr}$.

(2) $U_{n1}, \cdots, U_{nr}$ are linearly independent.

We shall define the standard monomials recursively, and prove in the next section that they have the desired properties (1) and (2).

DEFINITION. $x_1, \cdots, x_q$ are the standard monomials of degree one. If we have defined standard monomials of degrees $1, \cdots, n-1$, they are simply ordered in some way so that $u < v$ if degree $u <$ degree v. If degree $u = r$, degree $v = s$, and $r+s=n$, then $[u, v]$ is a standard monomial if both of the following conditions hold:

(S1) u and v are standard monomials and $u > v$.

(S2) If $u = [x, y]$ is the form of the standard monomial u, then

$v \geqq y$.

An element of L will be said to be in standard form if it is a linear combination of standard monomials.

3. The main theorem.

THEOREM 3.1. *The standard monomials form a basis for the free Lie ring L generated by $x_1, \cdots, x_q$.*

PROOF. The homogeneous expressions[2] of degree one are linear combinations of $x_1, \cdots, x_q$ and these are the standard monomials of degree one. Thus property (1) holds for degree $n=1$. Now suppose property (1) to hold for all degrees $1, \cdots, n-1$. We shall reduce $\lambda = \sum_k t_k[y_k, z_k]$, a homogeneous expression of degree n, to standard form by a canonical process which will be seen to leave λ unchanged if λ is in standard form.

First step: Let $y_k = \sum_i a_{ik}u_{ik}$, $z_k = \sum_j b_{jk}v_{jk}$ be the standard forms of y_k and z_k where the u's and v's are standard monomials. Put

$$\sum_k t_k[y_k, z_k] = \sum_{i,j,k} t_k a_{ik} b_{jk}[u_{ik}, v_{jk}]. \tag{3.1}$$

Second step: If u and v are standard monomials, put

$$[u, v] = 0 \quad \text{if } u = v, \tag{3.2.1}$$

$$[u, v] = -[v, u] \quad \text{if } u < v, \tag{3.2.2}$$

$$[u, v] = [u, v] \quad \text{if } u > v. \tag{3.2.3}$$

Third step: If $u > v$ are standard monomials and $u = [z, w]$ is the standard form of u, put

$$[u, v] = [z, w, v] \quad \text{if } v \geqq w, \tag{3.3.1}$$

$$[u, v] = -[w, v, z] + [z, v, w] \quad \text{if } v < w. \tag{3.3.2}$$

Fourth step: Return to the first step and repeat the processes until nothing but linear combinations of standard monomials of degree n remain.

---

[2] Throughout this section we may, with Witt, regard the free Lie ring L as a residue class ring of a free distributive, non-associative ring D. An element of L is a residue class of an ideal I in D. An "expression" is any representative of a residue class and the canonical process given here is a recursive method for finding a standard representative for each residue class. The first part of the proof consists in showing that there is a linear combination of standard monomials in every residue class, and the second part shows that there is only one. As an analogue, if we define the elements of a free group as classes of equivalent words, we must show that each class contains one and only one reduced word.

It will be necessary for us to show that the above process terminates. From the definition of the standard monomials it is clear that expressions in standard form are left unaltered by the canonical process, and that expressions left unaltered are in standard form.

If min (degree y, degree z) $>n/3$, then the process for $[y, z]$ will terminate with the second step. For we obtain monomials $[u, v]$ with $u>v$ and degree $v>n/3$, degree $u<2n/3$. And if $u=[z, w]$, then $z>w$ whence degree $w\leqq 1/2$ (degree u) $<n/3$. Hence from their degrees alone $v>w$ and (3.3.1) applies and $[u, v]$ is in standard form.

When the process does not terminate at the second step, then we go from a monomial $[u, v]$, $u>v$, where $u=[z, w]$ and $z>w>v$ to the monomials $[w, v, z]$ and $[z, v, w]$. Here w and z are standard monomials and $z>w>v$, while deg $[w, v]>$deg v, and deg $[z, v]>$deg v. Hence in applying the first step to $[w, v, z]$ and $[z, v, w]$ we get new expressions $[u', z]$, $[u'', w]$ with u', u'', z, and w all later than v in the ordering. Hence if the process continues long enough, we shall eventually reach monomials $[u', v']$ with min (deg u', deg v') $>n/3$, and as shown above the process will then terminate at the second step. Thus in all cases the canonical process ultimately terminates and yields an expression in standard form. Thus we have proved that the standard monomials have the first property required for a basis.

It is to be emphasized that the canonical process puts every expression in L into canonical form in a unique way. It will be necessary in proving the second property to show that different expressions for the same element of L lead to the same standard form. It has already been noted that the canonical process leaves unaltered any expression in standard form.

Since the relations (2.1) are of degree at least two, they cannot lead to any linear relation on $x_1, \cdots, x_q$, whence the second property for a basis holds for the standard monomials of degree one.

We shall complete the proof of the linear independence of the standard monomials by showing that any expression equal to zero in L has zero as its standard form. From the remarks of §2 we may confine our attention to homogeneous expressions. Now suppose the standard monomials of degrees $1, \cdots, n-1$ linearly independent and proceed by induction.

If h is any expression in L, let us write h^* for the standard form given for h by the canonical process. By our induction if $h=k$ are expressions of degree at most $n-1$ for the same element, then $h^*=k^*$. For expressions of degree n it is easy to verify from the process that $(-h)^*=-h^*$, that $(h+k)^*=h^*+k^*$, and that if $[h, k]$ is of degree n,

then $[h, k]^* = [h^*, k^*]^*$. Thus if $h = 0$ with h of degree less than n, and $[h, k]$ is of degree n, then $[h, k]^* = [0, k^*]^* = 0$.

If h is any expression of degree n which has the value 0 in L, then h is of the form $h = \sum[x, y] + \sum[R, R] + \sum([S, T] + [T, S]) + \sum([A, B, C] + [B, C, A] + [C, A, B])$, where in $\sum[x, y]$, x or y has the value zero. As noted above, the canonical process will yield zero for these terms. Hence we must show

$$[R, R]^* = 0, \tag{3.4.1}$$

$$[S, T]^* + [T, S]^* = 0, \tag{3.4.2}$$

$$[A, B, C]^* + [B, C, A]^* + [C, A, B]^* = 0. \tag{3.4.3}$$

From step 2 of the process, $[u, u]^*$ and $[u, v]^* + [v, u]^*$ are zero if u and v are standard monomials. This leads to immediate verification of (3.4.1) and (3.4.2).

The verification of (3.4.3) is more difficult. Suppose $A = A_1 + A_2$. Then $[A, B, C]^* = [[A, B]^*, C]^* = [[A_1, B]^* + [A_2, B]^*, C^*]^* = [[A_1, B]^*, C^*]^* + [[A_2, B]^*, C^*]^* = [A_1, B, C]^* + [A_2, B, C]^*$. Similarly $[A, B, C]^*$ is linear in its other two components. Thus we may reduce the verification of (3.4.3) to the case in which each of A, B, C is a standard monomial. If any two of A, B, C are equal, then (3.4.3) is trivial. From the cyclical symmetry of (3.4.3) suppose the notation such that $A > B$, $A > C$ in the ordering of the monomials. If necessary change the sign and interchange B and C so that $B > C$. Hence we need only verify (3.4.3) with standard monomials $A > B > C$.

If $[A, B]$ is a standard monomial, clearly $[A, B] > C$ and in the process for $[A, B, C]$ no alteration is involved in steps 1 or 2. In step 3, since $B > C$ we put $[A, B, C] = -[B, C, A] + [A, C, B]$. Hence $[A, B, C]^* + [B, C, A]^* + [C, A, B]^* = [A, C, B]^* + [C, A, B]^* = [0, B]^* = 0$. Note that if deg $B > 1/2$ deg A, then $[A, B]$ will be a standard monomial and the above argument will apply. As $A > B > C$, this will surely hold if deg $C > n/4$. We shall now use induction, assuming the validity of (3.4.3) for degree n for standard commutators $A' > B' > C'$ whenever $C' > C$, and for $C' = C$ whenever $B' > B$. Hence from the linearity of (3.4.3) in each of A, B, and C, our induction allows us to assume the validity of (3.4.3) for A'', B'', C'' when each of these is the sum of monomials all later than C, or with $C'' = C$ and A'', B'' the sum of monomials all later than B.

It remains to prove (3.4.3) with $A > B > C$ when $[A, B]$ is not a standard monomial. From (S2) this implies $A = [D, E]$ with $D > E > B$. Here, omitting the stars:

$$
\begin{aligned}
[A, B, C] = [D, E, B, C] &= -[E, B, D, C] + [D, B, E, C] \\
&= [D, C; E, B] - [E, B, C, D] \\
&\quad - [E, C; D, B] + [D, B, C, E] \\
&= [D, C; E, B] + [B, C, E, D] - [E, C, B, D] \\
&\quad + [D, B; E, C] - [B, C, D, E] + [D, C, B, E] \\
&= [D, C; E, B] + [B, C, E, D] - [E, C, B, D] \\
&\quad + [D, B; E, C] + [D, E; B, C] - [B, C, E, D] \\
&\quad + [D, C, B, E].
\end{aligned} \tag{3.5.1}
$$

Here since deg $[E, B] >$ deg B and $D > B$ we have applied the induction to replace $[E, B, D, C]$ by $-[D, C; E, B] + [E, B, C, D]$. Similar considerations apply elsewhere.

$$
\begin{aligned}
[B, C, A] &= [B, C; D, E] = -[D, E; B, C], \\
[C, A, B] &= -[A, C, B] = -[D, E, C, B] \\
&= [E, C, D, B] - [D, C, E, B] \\
&= -[D, B; E, C] + [E, C, B, D] - [D, C; E, B] \\
&\quad - [D, C, B, E].
\end{aligned} \tag{3.5.2, 3.5.3}
$$

Adding together (3.5.1), (2), and (3) we have

$$[A, B, C]^* + [B, C, A]^* + [C, A, B]^* = 0, \tag{3.5.4}$$

completing the proof of the theorem.

4. The theorem for higher commutators in groups. In the free group F with free generators $x_1, \cdots, x_q$, if we write $a^{-1}b^{-1}ab = (a, b)$ to define a commutator, there is a correspondence with elements in the free Lie ring L if we put (a, b) in correspondence with $[a, b]$. If F_n is the group generated by commutators of weight n and higher in F, then this correspondence, as Witt has shown, yields an isomorphism between the additive group of the homogeneous elements of degree n in L and the multiplicative group F_n/F_{n+1}. Thus we may apply the main theorem to F merely by translating our terms.

DEFINITION. In F the generators $x_1, \cdots, x_q$ are the standard commutators of weight one. Suppose that standard commutators of weights $1, \cdots, n-1$ have been defined and simply ordered so that $u < v$ if weight $u <$ weight v. Then (u, v) of weight n is a standard commutator if both the following conditions hold:

(S1) u, v are standard commutators and $u > v$,

(S2) If u is the commutator (z, w), then $v \geqq w$.

From Theorem 3.1 and the isomorphism established by Witt, we find immediately:

THEOREM 4.1. *The standard commutators of weight n are a basis of* F_n/F_{n+1}.

It will not be difficult for the reader to verify that the standard commutators are precisely those which arise in Philip Hall's collecting process given in [2]. Note that if (a, b) and c are commutators which arise in the collecting process, then it is clearly necessary for the existence of (a, b, c) in the collecting process that: (1) c is collected before (a, b); and (2) for (a, b) to exist when c is collected, either $c=b$ or c is collected after b.

BIBLIOGRAPHY

1. Garrett Birkhoff, *Representability of Lie algebras and Lie groups by matrices*, Ann. of Math. vol. 38 (1937) pp. 526–532.

2. Philip Hall, *A contribution to the theory of groups of prime power order*, Proc. London Math. Soc. vol. 36 (1934) pp. 29–95.

3. W. Magnus, *Über Beziehungen zwischen höheren Kommutatoren*, J. Reine Angew. Math. vol. 177 (1937) pp. 105–115.

4. E. Witt, *Treue Darstellung Liescher Ringe*, J. Reine Angew. Math. vol. 177 (1937) pp. 152–160.

OHIO STATE UNIVERSITY

INFINITE-DIMENSIONAL IRREDUCIBLE REPRESENTATIONS OF CERTAIN GROUPS

F. I. MAUTNER

Various considerations in the study of unitary representations of groups lead to the following question (cf. [1] and [2]):[1] Given a group G every one of whose elements has only a finite number of distinct conjugates; is every irreducible unitary representation of G finite-dimensional? Attempts have been made to prove that the answer is affirmative; in this note we give examples which settle the problem in the negative. These examples also show that the structure of the unitary representations of even countable discrete groups is more complicated than the so far existing theory (cf. [3], [4], and [5]) would lead one to expect. We shall come back to this point elsewhere.

Let F be any field with a finite number of elements $\alpha, \beta, \cdots$ and $\mathfrak{g}$ the group of all matrices

$$g = \begin{pmatrix} \alpha & \beta \\ 0 & 1 \end{pmatrix}$$

with α and β arbitrary elements of F, $\alpha \neq 0$. Let $\mathfrak{a}$ be the subgroup of matrices

$$\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix}$$

and $\mathfrak{b}$ the subgroup of matrices

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}.$$

Let $x(b)$ be an arbitrary character of the commutative group $\mathfrak{b}$, that is, $b \to x(b)$ is a homomorphism of $\mathfrak{b}$ to the multiplicative group of complex numbers of absolute value one. We shall need the following lemma for which F need not yet be finite.

LEMMA 1. *If there exists an element a of $\mathfrak{a}$, $a \neq$ identity of $\mathfrak{a}$, such that $x(aba^{-1}) = x(b)$ for all elements b of $\mathfrak{b}$, then $x(b) = 1$ for all b.*

PROOF. The proof is obvious.

Let G be the group all infinite sequences $(g_1, g_2, \cdots)$ where each $g_n \in \mathfrak{g}$ and each sequence has only a finite number of elements not

Presented to the Society, October 29, 1949; received by the editors July 18, 1949.

[1] Numbers in brackets refer to the references cited at the end of the paper.

equal to the identity element e of $\mathfrak{g}$; elements of G are to be multiplied componentwise. Then it is clear that each element of G has only a finite number of distinct conjugates.

Let A be the subgroup of those elements $A=(g_1, g_2, \cdots)$ of G for which each $g_n \in \mathfrak{a}$ and similarly B be the subgroup of those elements B all of whose components are in $\mathfrak{b}$. Since every element g of $\mathfrak{g}$ is a unique product $g=ba$ with $b\in\mathfrak{b}$, $a\in\mathfrak{a}$, therefore every element G of G can be written uniquely as a product $G=BA$ with $B\in B$, $A\in A$.

For every character $X(B)$ of the commutative group B we now define a unitary representation of the group G in a Hilbert space H as follows. Let H consist of all those complex-valued functions $f(A)$ defined on A for which $\|f\|^2=\sum_{A\in A}|f(A)|^2<\infty$. With every element A_0 of A we associate the unitary operator $U(A_0)\colon f(A)\to f(AA_0)$, and with every $B_0\in B$ the operator $U(B_0)\colon f(A)\to X(AB_0A^{-1})f(A)$. If $G=BA$ let $U(G)=U(B)U(A)$, then it is immediately verified that $G\to U(G)$ is a unitary representation of G, that is, a homomorphism of G into the multiplicative group of unitary operators of H.

Let $X(B)=\prod_{n=1}^{\infty} x_n(b_n)$ where $x_n(b)$ is for each n a character of $\mathfrak{b}$ and b_n the nth component of B.

LEMMA 2. *If $X(B)$ is such that $x_n(b)$ is not the constant character for any n, then the above representation is irreducible, that is, there are no proper closed linear subspaces of H invariant under all the $U(G)$.*

PROOF. Suppose there is a bounded linear operator C in H which commutes with all $U(G)$. Then $CU(A)=U(A)C$ for all A implies by Lemma 5.3.2 of [6] that there exists a complex-valued function $c(A)$ defined on A such that

$$(Cf)(A)=\sum_{A'\in A} c(AA'^{-1})f(A').$$

Now $U(B)C=CU(B)$ for all $B\in B$ implies

(1) $\quad X(ABA^{-1})c(A)=X(B)c(A)$ for all $A\in A$ and $B\in B$.

If there is an A such that $X(ABA^{-1})=X(B)$ for all B then $x_n(a_nba_n^{-1})=x_n(b)$ for all $b\in\mathfrak{b}$, a_n being the nth component of this A. By hypothesis $x_n(b)$ is not the constant character of $\mathfrak{b}$, therefore Lemma 1 implies that $a_n=e$ for all n, hence $A=E=$identity element of A. Thus for any $A\neq E$ there exists an element B such that $X(ABA^{-1})\neq X(B)$. Therefore equation (1) implies $c(A)=0$ for all $A\neq E$. Hence C is a scalar multiple of the identity transformation of H.

Thus we have constructed infinite-dimensional irreducible unitary representations of G in a completely elementary manner.

References

1. R. Godement, *Analyse harmonique dans les groupes centraux* II, C. R. Acad. Sci. Paris vol. 225 (1947) pp. 221–223.

2. F. I. Mautner, *Structure of the regular representation of certain discrete groups*, To appear in Duke Math. J.

3. ———, *The completeness of the irreducible unitary representations of a locally compact group*, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 52–54.

4. ———, *Unitary representations of locally compact groups* I, Ann. of Math. vol. 51 (1950) pp. 1–25.

5. J. von Neumann, *On rings of operators. Reduction theory*, Ann. of Math. vol. 50 (1949) pp. 401–485.

6. J. von Neumann and F. J. Murray, *On rings of operators* IV, Ann. of Math. vol. 44 (1943) pp. 716–808.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

A NOTE ON THE CHARACTERISTIC POLYNOMIALS OF CERTAIN MATRICES

WILLIAM T. REID

The purpose of this note is to establish the following theorem on matrices with elements in an arbitrary field $\mathfrak{F}$.[1] In particular, I denotes the $n\times n$ identity matrix; 0 is used for the zero matrix, square or rectangular, its dimensions being clear in each case from the context. The determinant of an $n\times n$ matrix $M(\lambda)$ with elements in $\mathfrak{F}(\lambda)$, the ring of polynomials in λ with coefficients in $\mathfrak{F}$, is denoted by $|M(\lambda)|$; in particular, if M is an $n\times n$ matrix with elements in $\mathfrak{F}$, then $|\lambda I-M|$ is the characteristic polynomial of M.

THEOREM. *If A is an $n\times m$ matrix, and D is an $n\times n$ matrix, then in order that $|\lambda I-AB|=|\lambda I-AB-D|$ for arbitrary $m\times n$ matrices B it is necessary and sufficient that D be nilpotent and $DA=0$. In particular, if C is an $m\times n$ matrix, then in order that $|\lambda I-AB|=|\lambda I-A(B+C)|$ for arbitrary $m\times n$ matrices B it is necessary and sufficient that $ACA=0$.*

Presented to the Society, September 2, 1949; received by the editors April 5, 1949 and, in revised form, July 23, 1949.

[1] In the original manuscript of this paper attention was restricted to $\mathfrak{F}$ the field of complex numbers. The proof there given of the fact that the nilpotency of D and $DA=0$ imply $|\lambda I-AB|=|\lambda I-AB-D|$, for arbitrary B, was valid for arbitrary fields $\mathfrak{F}$. The proof here given of the necessity of $DA=0$ for $\mathfrak{F}$ an arbitrary field was suggested by the referee.

An alternate proof of the sufficiency part of the statement in the last sentence of the theorem has been obtained recently by Parker,[2] in generalizing an earlier result of A. Brauer.[3]

The principal tool used in the proof of the above theorem is the fact that a matrix D is nilpotent if and only if $\lambda^n = |\lambda I - D|$. That $\lambda^n = |\lambda I - D|$ implies the nilpotency of D follows from the Cayley-Hamilton theorem. On the other hand, if D is nilpotent of index s, then $|\lambda I - D^r| = \lambda^n$ for $r \geqq s$, while if $\lambda^n = |\lambda I - D^r|$ for $r \geqq t+1 > 1$, then $|\lambda I - D^t| \cdot |\lambda I + D^t| = |\lambda^2 I - D^{2t}| = |\lambda^2 I| = \lambda^{2n}$, so that $|\lambda I - D^t| = \lambda^n$, and by induction $|\lambda I - D| = \lambda^n$.

The sufficiency part of the statement in the first sentence of the above theorem is established by noting that if D is nilpotent and $DA = 0$, then $\lambda^n |\lambda I - AB| = |(\lambda I - D)(\lambda I - AB)| = |\lambda^2 I - \lambda(AB + D)| = \lambda^n |\lambda I - AB - D|$, and $|\lambda I - AB| = |\lambda I - AB - D|$ for arbitrary $m \times n$ matrices B.

On the other hand, suppose that $|\lambda I - AB| = |\lambda I - AB - D|$ for all $m \times n$ matrices B. The choice $B = 0$ gives $|\lambda I - D| = \lambda^n$, and hence the nilpotency of D. Then $|\lambda^2 I - \lambda(AB + D)| = \lambda^n |\lambda I - (AB + D)| = |\lambda I - D| \cdot |\lambda I - AB| = |\lambda^2 I - \lambda(AB + D) + DAB|$ for arbitrary $m \times n$ matrices B. Now if $DA = K = \|K_{i\beta}\|$ $(i = 1, \cdots, n; \beta = 1, \cdots, m)$ has a nonzero element K_{pq}, then the choice $B_0 = \|\delta_{\alpha q} \delta_{pj} K_{pq}^{-1}\|$ $(\alpha = 1, \cdots, m; j = 1, \cdots, n)$ gives $DAB_0 = G = \|G_{ij}\|$ $(i, j = 1, \cdots, n)$ with $G_{ij} = 0$ if $j \neq p$, $G_{pp} = 1$. If $AB_0 + D = H = \|H_{ij}\|$ $(i, j = 1, \cdots, n)$, then from the above, $|\lambda^2 I - \lambda H| = |\lambda^2 - \lambda H + G|$. This is impossible, however, since $|\lambda^2 I - \lambda H + G| = |\lambda^2 I - \lambda H| + |R(\lambda)|$, where $R_{ij}(\lambda) = \lambda^2 \delta_{ij} - \lambda H_{ij}$ if $j \neq p$, $R_{ip}(\lambda) = G_{ip}$, and consequently $|R(\lambda)|$ is a nonzero polynomial λ^{2n-2} + terms of lower degree. Hence $DA = 0$, and the statement of the first sentence of the theorem is proved.

The second sentence of the theorem follows from the first part, together with the remark that if $D = AC$, then $0 = DA = ACA$ implies $0 = DAC = D^2$ so that D is nilpotent.

NORTHWESTERN UNIVERSITY

[2] W. V. Parker, *On the characteristic equations of certain matrices*, Bull. Amer. Math. Soc. vol. 55 (1949) pp. 115–116.

[3] Alfred Brauer, *On the characteristic equations of certain matrices*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 605–607.

ON THE MATRIC EQUATION $X^2+AX+XB+C=0$

WILLIAM E. ROTH

The equation

$$X^2 + AX + XB + C = 0, \tag{1}$$

where A, B, and C are $n \times n$ matrices with elements in F, the field of complex numbers or a subfield thereof will be solved for $n \times n$ matrices X with elements in F. Throughout the discussion which follows, the elements of all matrices and the coefficients of all polynomials which arise will be in the field F even when this is not specifically stated. Moreover the similarity of matrices and the reducibility of polynomials will be valid under the rational operations of F.

It should be remarked that equation (1) becomes unilateral if X be replaced by $Y-A$ or by $Z-B$ and in that form has been studied heretofore.[1] This fact was noted by the writer only after the essentials of the present note had been discovered.

1. **Necessary conditions.** Let

$$R = \begin{pmatrix} -B, & I \\ -C, & A \end{pmatrix}$$

and let X be any solution of (1) with elements in F;[2] then

$$\begin{pmatrix} X, & I \\ I, & 0 \end{pmatrix} R \begin{pmatrix} 0, & I \\ I, & -X \end{pmatrix} = \begin{pmatrix} X+A, & -X^2 - AX - XB - C \\ I, & -X - B \end{pmatrix} = \begin{pmatrix} X+A, & 0 \\ I, & -X-B \end{pmatrix}. \tag{2}$$

Thus the matrices

$$R \quad \text{and} \quad \begin{pmatrix} X+A, & 0 \\ I, & -X-B \end{pmatrix}$$

are similar, and $|R-\lambda I|$ is reducible to polynomials $f(\lambda)$ and $g(\lambda)$,

Presented to the Society, September 1, 1949; received by the editors June 29, 1949.

[1] Ingraham, *Rational methods in matric equations*, Bull. Amer. Math. Soc. vol. 47 (1941) pp. 61–70; Roth, *On the unilateral equation in matrices*, Trans. Amer. Math. Soc. vol. 32 (1930) pp. 61–80. Either paper cites further references.

[2] The letter I will represent unit matrices of order n or $2n$ to agree with that of other matrices in the same expression. Thus in R above I is of order n.

namely the characteristic polynomials of $X+A$ and $-X-B$, respectively, with coefficients in F. The following theorem, which is implied by the above, will be a guide in the quest of solutions of the given equation.

THEOREM I. *If equation* (1) *has a solution with elements in* F, *then there exists at least one pair of polynomials* $f_\alpha(\lambda)$ *of degree* $\alpha \leqq n$ *and* $g_\beta(\lambda)$ *of degree* $\beta \leqq n$ *with coefficients in* F *such that* $f_\alpha(R)g_\beta(R)=0$, *where* $f_\alpha(\lambda)g_\beta(\lambda)$ *is not necessarily the minimum polynomial satisfied by* R *but is a divisor of* $|R-\lambda I|$ *and such that* $f_\alpha(X+A)=0$ *and* $g_\beta(-X-B)=0$.

Polynomials $f_\alpha(\lambda)$ and $g_\beta(\lambda)$, of degree $\alpha \leqq n$ and $\beta \leqq n$ respectively, such that $f_\alpha(\lambda)g_\beta(\lambda)$ is a divisor of $|R-\lambda I|$ and a multiple of the minimum polynomial satisfied by R will be designated as admissible polynomials.

Let $f_\alpha(\lambda)$ be an admissible polynomial such that $f_\alpha(X+A)=0$, where X is a solution of (1) and let

$$f_\alpha(R) = \begin{pmatrix} U, & M \\ V, & N \end{pmatrix},$$

where U, V, M, and N are polynomials in the matrices A, B, and C; hence according to (2) we have

$$\begin{pmatrix} X, & I \\ I, & 0 \end{pmatrix} f_\alpha(R) \begin{pmatrix} 0, & I \\ I, & -X \end{pmatrix} = \begin{pmatrix} f_\alpha(X+A), & 0 \\ *, & f_\alpha(-X-B) \end{pmatrix}$$

$$= \begin{pmatrix} XM+N, & XU+V-(XM+N)X \\ M, & U-MX \end{pmatrix};$$

and consequently the equations,

$$XM+N=0, \qquad XU+V=0, \tag{3}$$

are simultaneously satisfied whether M be singular or not, and

$$(X, I)f_\alpha(R) = (X, I)\begin{pmatrix} U, & M \\ V, & N \end{pmatrix} = 0. \tag{4}$$

Now let

$$f_\alpha(R)\begin{pmatrix} I \\ -X_1 \end{pmatrix} = \begin{pmatrix} U, & M \\ V, & N \end{pmatrix}\begin{pmatrix} I \\ -X_1 \end{pmatrix} = 0,$$

and M be nonsingular.[3] Then $f_\alpha(R)$ and M have the same rank n

[3] It can be shown that M is singular only if, but not necessarily if, $f_\alpha(\lambda)$ and $g_\beta(\lambda)$ have a common factor.

and a unique solution of this equation exists such that

$$U - MX_1 = 0, \qquad V - NX_1 = 0. \tag{5}$$

We shall show that X_1 so obtained is also a solution of (1) and that

$$X_1 = -M^{-1}(X + B)M - A, \tag{6}$$

where X is a solution of (1) and as such satisfies (4).

Because R and $f_\alpha(R)$ are commutative, we readily obtain the relations:

$$\begin{aligned} N &= U + MA + BM, \\ V &= BU - UB - MC = AN - NA - CM, \end{aligned} \tag{7}$$

and by (5)

$$\begin{aligned} V - NX_1 &= BU - UB - MC - (U + MA + BM)X_1 \\ &= BMX_1 - MX_1B - MC - (MX_1 + MA + BM)X_1 \\ &= -M(X_1^2 + AX_1 + X_1B + C) = 0. \end{aligned}$$

Now M is nonsingular and X_1 is therefore a solution of (1). Equation (6) follows from (3), (5), and the first equation of (7).

The solutions X and X_1 of equation (1) are such that $f_\alpha(X+A)=0$ and $f_\alpha(-X_1-B)=0$ and arise in case $f_\alpha(R)$ and

$$\begin{pmatrix} 0, & 0 \\ M, & 0 \end{pmatrix}$$

are equivalent under elementary transformations since

$$\begin{pmatrix} X, & I \\ I, & 0 \end{pmatrix} f_\alpha(R) \begin{pmatrix} 0, & I \\ I, & -X_1 \end{pmatrix} = \begin{pmatrix} X, & I \\ I, & 0 \end{pmatrix} \begin{pmatrix} U, & M \\ V, & N \end{pmatrix} \begin{pmatrix} 0, & I \\ I, & -X_1 \end{pmatrix}$$
$$= \begin{pmatrix} 0, & 0 \\ M, & 0 \end{pmatrix}.$$

The following theorem is proved.

THEOREM II. *If X is a solution of equation* (1) *with elements in F, then there exists a polynomial $f_\alpha(\lambda)$ of degree $\alpha \leqq n$ with coefficients in F such that equation* (4) *is satisfied. If in addition M is nonsingular, $X_1 = -M^{-1}(X+B)M - A$ is also a solution of* (1) *with elements in F.*

2. **The converse problem.** We shall state the theorem:

THEOREM III. *If $f_\alpha(\lambda)$ is a polynomial of degree $\alpha \leqq n$ with coefficients in F such that*

$$f_a(R) = \begin{pmatrix} U, & M \\ V, & N \end{pmatrix}$$

is of rank n and if M is nonsingular, then X such that $(X, I)f_a(R)=0$ *is a solution of equation* (1) *with elements in F and* $X_1 = -M^{-1}(X+B)M - A$ *is likewise such a solution.*

The proof of this theorem is entirely analogous to that above by which X_1 was shown to be a solution of (1) and will not be given here. Though Theorem III holds for any polynomial $f_a(\lambda)$ of whatever degree in λ; those of lowest degree which may satisfy the hypotheses will be found only among the admissible polynomials defined above.

Addendum. The referee's suggestion that the results above might be extended to the equation

$$XDX + AX + XB + C = 0,$$

where D is nonsingular, was anticipated by Professor James H. Bell in a letter to the writer dated March 3, 1949. This generalization appeared of slight significance because the product of this equation on the right (on the left) by D can readily be transformed to (1) in case D is nonsingular. However, the writer has subsequently noted that if I be replaced by D in R and consequently if $X+A$ be replaced by $XD+A$ and $X+B$ by $DX+B$, as suggested by the referee, the content of the above paper after appropriate minor changes is still valid even when D is singular. Moreover, in case D is singular, this equation is not transformable to the unilateral form and consequently is of particular interest. The writer wishes to express his indebtedness to Professor Bell and to the referee.

UNIVERSITY OF TULSA

SOME REMARKS ON A THEOREM OF ERDÖS CONCERNING ASYMPTOTIC DENSITY

HAROLD N. SHAPIRO

In [1] Erdös proves the following:[1]

THEOREM. *If A and B are non-empty sets of positive integers, of asymptotic densities α, β, respectively; and if furthermore*

$$\alpha + \beta \leqq 1, \tag{1}$$

$$1 \in B, \tag{2}$$

$$\alpha \geqq \beta, \tag{3}$$

then for the asymptotic density γ of the sum set $C=A+B$ (consisting of the integers of A, B, and the sums $a+b$, $a\in A$, $b\in B$) we have

$$\gamma \geqq \alpha + \beta/2. \tag{4}$$

More precisely, at least one of the sets

$$\{a+b\}, \quad \{a, a+1\} \qquad a \in A, b \in B \tag{5}$$

must have asymptotic density greater than or equal to $\alpha+\beta/2$.

Due to the asymmetry introduced by the hypothesis (2), the above theorem leaves open the case where $\beta>\alpha$. Actually, the entire theorem as it stands holds without hypothesis (3). This results from the fact that Erdös' proof of the theorem given in [1] uses only the condition $\alpha>\beta/2$. In the contrary case, $2\alpha\leqq\beta$, if we let a_1 be any fixed integer of A, the set $\{b+a_1\}$, $b\in B$, has asymptotic density $\beta\geqq\alpha+\beta/2$; and the theorem follows.

In this note it is proposed to give (using only hypotheses (1) and (2)) a short proof of (4) which does not however yield the more precise information concerning the sets of (5). This proof is based upon the (α, β) hypothesis for Schnirelman density, proved in [2], [3].

If either α or β equals 0, (4) follows trivially. Hence we may assume that both α and β are not 0. Since α is the asymptotic density of A, given any ϵ, $\alpha>\epsilon>0$, we can find a largest integer m such that

$$A(m) < (\alpha - \epsilon)m. \tag{6}$$

(If $A(z)\geqq(\alpha-\epsilon)z$ for all $z\geqq 1$, we take $m=0$.) We then consider the

Received by the editors July 25, 1949.

[1] The definitions of all terms used may be found in [1]. Numbers in brackets refer to the references cited at the end of the paper.

positive terms of the sequence a_i-m, $a_i\in A$ and form the set

$$A_\epsilon = \{a_i - m\}.$$

We then have for $x>0$,

$$\begin{aligned} A_\epsilon(x) &= A(x+m) - A(m) \\ &\geqq (\alpha - \epsilon)x \end{aligned}$$

so that the Schnirelman density of the set A_ϵ is not less than $\alpha-\epsilon$. In particular, then, $1\in A_\epsilon$, whence $m+1\in A$.

Similarly, we construct a

$$B_\epsilon = \{b_j - l\}, \qquad b_j \in B, l+1 \in B,$$

which has Schnirelman density not less than $\beta-\epsilon$. Then, by the (α, β) hypothesis, the Schnirelman density of the set

$$C_\epsilon = \{a_i - m, b_j - l, a_i + b_j - m - l\}$$

is not less than $\alpha+\beta-2\epsilon$. Thus for any $x>0$,

$$C_\epsilon(x) \geqq (\alpha + \beta - 2\epsilon)x.$$

Adding $m+l+1$ to the sequence C_ϵ we obtain

$$\bar{C}_\epsilon = \{a_i + l + 1, b_j + m + 1, a_i + b_j + 1\}$$

where for large x,

$$\bar{C}_\epsilon(x) \geqq (\alpha + \beta - 2\epsilon)x - (m + l + 1).$$

However, since $l+1\in B$, $m+1\in A$,

$$\bar{C}_\epsilon \subset \{a_i + b_j, a_i + b_j + 1\} = \bar{C}.$$

Hence for large x,

$$\bar{C}(x) \geqq (\alpha + \beta - 2\epsilon)x - (m + l + 1).$$

Next, letting $\nu=$the number of $a_i+b_j\leqq x$ such that $a_i+b_j+1 \notin \{a_i+b_j\}$, we consider two cases:

Case 1. $\nu\leqq(\beta/2)x$. Then the number of $\{a_i+b_j\}$ less than or equal to x is greater than or equal to $(\alpha+\beta/2-2\epsilon)x-(m+l+1)$.

Case 2. $\nu>(\beta/2)x$. If $a_i+b_j+1\notin\{a_i+b_j\}$, then $a_i+b_j\notin A$ (since $1\in B$). Hence in this case the set $\{a_i, a_i+b_j\}$ contains (for large x) more than $(\alpha+\beta/2-\epsilon)x$ integers less than or equal to x.

Thus in any case we obtain that for x large enough the number of integers not greater than x in the set $\{a_i, a_i+b_j\}$ is greater than $(\alpha+\beta/2-2\epsilon)x-(m+l+1)$, which suffices to establish (4).

References

1. P. Erdös, *On the asymptotic density of the sum of two sequences*, Ann. of Math. vol. 43 (1942) pp. 65–68.
2. H. B. Mann, *A proof of the fundamental theorem on the density of sums of sets of positive integers*, Ann. of Math. vol. 43 (1942) pp. 523–527.
3. Emil Artin and Peter Scherk, *On the sum of two sets of integers*, Ann. of Math. vol. 44 (1943) pp. 138–142.

NEW YORK UNIVERSITY

CERTAIN BEST POSSIBLE RESULTS IN THE THEORY OF SCHNIRELMANN DENSITY

BENJAMIN LEPSON[1]

Let A be a set of distinct non-negative integers, and $A(n)$ the number of positive integers not greater than n in A. Then the Schnirelmann density α of A, referred to below simply as density, is defined as

$$\alpha = \text{g.l.b.}\ \frac{A(n)}{n}.$$

If A and B are two such sets, the set $C=A+B$ consists of all distinct sums $a+b$ with a in A and b in B. The densities of A, B, and C will be denoted by α, β, and γ respectively.

The $\alpha+\beta$ hypothesis, first proved by Mann [3],[2] states that, if A and B each contain 0, then

$$\gamma \geqq \min(1, \alpha+\beta). \tag{1}$$

Erdös and Niven [2] state that (1) is "a best possible result" without giving details. It is shown below that, under the above conditions, and for any pair (α, β), there exist sets for which the equality sign holds in (1).

THEOREM 1. *Let α and β be any real numbers such that*

$$0 \leqq \alpha \leqq 1 \quad \text{and} \quad 0 \leqq \beta \leqq 1.$$

Then there exist sets A and B, each containing 0 and of densities α and

Presented to the Society, October 29, 1949; received by the editors August 12, 1949.

[1] Atomic Energy Commission pre-doctoral fellow.

[2] Numbers in brackets refer to the references cited at the end of the paper.

β respectively, such that the density γ of C is given by

$$\gamma = \min(1, \alpha + \beta). \tag{2}$$

PROOF. For every $n>0$, let a_n be the least positive integer not greater than $\alpha \cdot n!$. Then

$$0 \leqq a_{n+1} - a_n \leqq (n+1)! - n!. \tag{3}$$

Define A as the set consisting of 0, 1, and each of the (possibly vanishing) blocks $n!+1$, $n!+2$, $\cdots$, $n!+a_{n+1}-a_n$. By (3), no two blocks have an element in common; therefore

$$\frac{A(n!)}{n!} = \frac{1 + \sum_{j=1}^{n-1} (a_{j+1} - a_j)}{n!} = \frac{a_n}{n!} \geqq \alpha.$$

Since

$$\lim_{n\to\infty} \frac{a_n}{n!} = \alpha$$

and

$$\frac{A(n)}{n} \geqq \frac{A(m!)}{m!} \geqq \alpha$$

for some m depending upon n, we see that the density of A is α. Define b_n and B similarly in terms of β.

Let $C=A+B$ have density γ, and let $c \leqq n!$ belong to C. Then $c=a+b$ where $a \leqq n!$ and $b \leqq n!$, so that, from the construction of A and B, $a \leqq (n-1)!+a_n-a_{n-1}$ and $b \leqq (n-1)!+b_n-b_{n-1}$. Therefore

$$\begin{aligned} c &\leqq 2(n-1)! + a_n + b_n - a_{n-1} - b_{n-1} \\ &\leqq 2(n-1)! + a_n + b_n \end{aligned}$$

which implies

$$C(n!) \leqq 2(n-1)! + a_n + b_n \tag{4}$$

from which it follows that

$$\lim_{n\to\infty} \frac{C(n!)}{n!} \leqq \alpha + \beta \tag{5}$$

and finally

$$\gamma \leqq \alpha + \beta. \tag{6}$$

Combining (1) and (6), we have (2).[3]

One may ask whether it is possible to weaken the condition that A and B both contain 0 and still to have (1). In this connection, we have the following result.

THEOREM 2. *Let E and F be finite sets of distinct non-negative integers. If, whenever A contains E and B contains F, (1) holds, then E and F both contain 0.*

PROOF. Suppose that E does not contain 0. Let N be the largest integer in E or F. Define A as the set of positive integers congruent to any of the integers $1, 2, \cdots, N \pmod{2N+1}$ and B as the set of non-negative integers congruent to any of the integers $0, 1, 2, \cdots, N \pmod{2N+1}$. Then

$$\alpha = \frac{N}{2N+1}, \qquad \beta = \frac{1}{2}, \qquad \gamma = \frac{2N}{2N+1}$$

so that

$$\gamma < \alpha + \beta < 1$$

which contradicts (1). Similarly, F contains 0.

Finally, we note that (1) cannot be improved by requiring A and B to contain certain finite sets in addition to 0. This follows immediately from Theorem 1, with the observations that the densities of the sets A and B used in the proof would be unchanged by the addition of finite sets, and that (4) would remain valid for sufficiently large n, implying (5) and (6).

REFERENCES

1. E. Artin and P. Sherk, *On the sum of two sets of integers*, Ann. of Math. vol. 44 (1943) pp. 138–142.

2. P. Erdös and I. Niven, *The $\alpha+\beta$ hypothesis and related problems*, Amer. Math. Monthly vol. 53 (1946) pp. 314–317.

3. H. B. Mann, *A proof of the fundamental theorem on the density of sums of sets of positive integers*, Ann. of Math. vol. 43 (1942) pp. 523–527.

COLUMBIA UNIVERSITY

[3] It is possible, by a more detailed argument, to prove (2) directly without the use of (1).

A NOTE ON ISOTHERMAL NETS

V. G. GROVE

1. **Introduction.** It is the purpose of this paper to give a geometrical characterization of isothermally conjugate nets. This characterization is not dependent on the axis or ray curves of the net, and therefore not on the notation of conjugacy of the axis congruence nor on the ray congruence being harmonic to the sustaining surface. The characterization is made in terms of a property described by Bell [1][1] of a congruence being *central to the surface S*. A family of congruences associated with the tangent to a curve of the conjugate net is derived. The condition that the net be isothermally conjugate is expressed in terms of the property of any congruence of the family being central to S. The notion is extended to a characterization of isothermally orthogonal nets.

To fix the notation, let the coordinates (x^1, x^2, x^3, x^4) of a generic point x of the surface S and the parameters u, v be so chosen that they satisfy the following system of differential equations [3]

$$(1.1) \qquad x_{uu} = \theta_u x_u + \beta x_v + px, \quad x_{vv} = \gamma x_u + \theta_v x_v + qx, \quad \theta = \log R.$$

The lines determined by the points x_u, x_v and by the points x, x_{uv} have been called the R-harmonic and R-conjugate lines respectively.

Let r and s be two points on the asymptotic tangents to the curves $v=$const., $u=$const. respectively. The coordinates r, s of these points are given by the expressions

$$(1.2) \qquad r = x_u - bx, \qquad s = x_v - ax.$$

It is readily shown that as x generates the curve $u=$const. on S, the point r generates a curve whose tangent is coplanar with the tangent to the locus of s as x generates $v=$const. if and only if

$$(1.3) \qquad a_u - b_v = 0.$$

The congruence Γ_2 of lines (r, s) is therefore central to S if and only if (1.3) is satisfied. If Γ_2 is central to S, then the curves corresponding to the developables of Γ_2 either form a conjugate net, coincide with asymptotic curves, or are indeterminate [5]. A similar statement holds for the reciprocal congruence Γ_1 of Γ_2.

2. **Isothermally conjugate nets.** Let the differential equation of a

Presented to the Society, September 1, 1949; received by the editors July 3, 1949.

[1] Numbers in brackets refer to the references cited at the end of the paper.

one-parameter family of curves C_λ on S be

$$Mdu - Ndv = 0, \qquad \lambda = M/N, \tag{2.1}$$

this family being neither of the families of asymptotic curves. Let y be a point on the tangent to that curve of C_λ through x, but not on the R-harmonic line. The coordinates of y have the form

$$y = Sx + Nx_u + Mx_v, \qquad MNS \neq 0.$$

The projections r, s of y from the points x_v and x_u on the asymptotic tangents have coordinates given by the expressions

$$r = x_u + Sx/N, \qquad s = x_v + Sx/M. \tag{2.2}$$

We have called these points the integral generating points [4]. However, in this paper we do not insist that they be invariant.

The tangent to the locus of $s(r)$ as x generates the curve $v=\text{const.}$ ($u=\text{const.}$) intersects the plane determined by the tangent to $u=\text{const.}$ ($v=\text{const.}$) and the R-conjugate line in a point whose projection from x_{uv} on the tangent plane is the point ρ (σ) whose coordinates are given by the expression [4]

$$\rho = x_u + \frac{\partial}{\partial u}\log\left(\frac{S}{M}\right)x, \qquad \sigma = x_v + \frac{\partial}{\partial v}\log\left(\frac{S}{N}\right)x. \tag{2.3}$$

If we define points ρ_h, σ_h so that one of the cross ratios of x, x_u, ρ, ρ_h, and of x, x_v, σ, σ_h, is h, the coordinates of these points are given by

$$\rho_h = x_u + h\frac{\partial}{\partial u}\log\frac{S}{M}\,x, \quad \sigma_h = x_v + h\frac{\partial}{\partial v}\log\frac{S}{N}\,x. \tag{2.4}$$

We shall call these points the *differential generating points*, and the line determined by them the *differential generating line*. A similar appellation will be used for the congruence of these lines.

From (1.3) it follows that *the differential generating congruence determined by* (2.4) *is central to S if and only if*

$$\frac{\partial^2}{\partial u\partial v}\log\left(\frac{M}{N}\right) = 0, \tag{2.5}$$

that is, if and only if the conjugate net

$$M^2du^2 - N^2dv^2 = 0 \tag{2.6}$$

is isothermally conjugate.

We shall say that the curves (2.1) have *the isothermal property* if (2.5) is satisfied. Hence if one family of curves of a conjugate net has

the isothermal property, the other has, and all curves of the pencil of conjugate nets whose base net is the given net have the property. An immediate application of this notion gives the following theorem. *A surface is isothermally asymptotic if and only if any curve of he Darboux-Segre pencil of conjugate nets has the isothermal property.*

3. **Conclusion.** Suppose a given conjugate net defined by (2.6) is isothermally conjugate. By a transformation of independent variables, we may make $M=N$. Equations (2.4) may be written in the form

$$(3.1) \qquad \rho_h = x_u - \frac{\partial}{\partial u}\log Ax, \qquad \sigma_h = x_v - \frac{\partial}{\partial v}\log Ax$$

wherein

$$A = (M/S)^h.$$

The curves on S corresponding to the developables of the differential generating congruence [5] are defined by the differential equation

$$Pdu^2 - Qdv^2 = 0,$$

wherein

$$P = A_{uu} - (\theta_u A_u + \beta A_v + pA),$$
$$Q = A_{vv} - (\gamma A_u + \theta_v A_v + qA).$$

It follows that *any solution* A *of* (1.1) *gives a congruence of lines* Γ_2 *whose developables are indeterminate.*

Similarly the developables of the reciprocal congruence Γ_1 of Γ_2 with respect to the quadrices of Darboux are indeterminate if and only if A is a solution of

$$A_{uu} = \theta_u A_u - \beta A_v + (p + \beta\theta_v + \beta_v)A,$$
$$A_{vv} = -\gamma A_u + \theta_v A_v + (q + \gamma\theta_u + \gamma_u)A.$$

If both congruences Γ_1, Γ_2 *have indeterminate developables,* S *is necessarily isothermally asymptotic.*

We now extend the characterization of isothermal conjugacy of a net to a characterization of isothermally orthogonal nets. Let the surface be referred to the orthogonal net as parametric. Let a one-parameter family of curves be given by (2.1). The coordinates y^i $(i=1, 2, 3)$ of any point y on the tangent to this curve have the form

$$y^i = x^i + (Nx^i_u + Mx^i_v)/S, \qquad i = 1, 2, 3.$$

The orthogonal projections r, s of y on the tangents to the curve of the net have coordinates given by the expressions

$$r^i = x^i + \frac{N}{S} x_u^i, \qquad S^i = x^i + \frac{M}{S} x_v^i. \tag{3.2}$$

As x generates the curve $v=\text{const.}$, s generates a curve whose tangent has the following direction numbers

$$s_u^i = x_u^i + \left(\frac{M}{S}\right)_u x_v^i + \frac{M}{S} x_{uv}^i. \tag{3.3}$$

This tangent intersects the plane determined by the tangent to $v=\text{const.}$ and Green's R-reciprocal line [2] of the ideal line in the tangent plane in the point whose coordinates are

$$x^i - \lambda\left(x_u^i + \frac{M}{S} x_{uv}^i\right), \qquad \frac{1}{\lambda} = \frac{\partial}{\partial u} \log\left(\frac{M}{S}\right).$$

If we project on the tangent plane to S this latter point by a line parallel to this R-reciprocal, a point ρ is found whose coordinates are given by the expression

$$\rho^i = x^i - \lambda x_u^i. \tag{3.4}$$

In a similar manner a point σ whose coordinates σ^i are given by the expression

$$\sigma^i = x^i - \mu x_v^i, \qquad \frac{1}{\mu} = \frac{\partial}{\partial v} \log\left(\frac{N}{S}\right) \tag{3.5}$$

may be defined. We shall say that the congruence of lines $(\rho\sigma)$ *is central* to S with respect to the given orthogonal parametric net if the tangent to the locus of ρ as x generates $u=\text{const.}$ intersects the tangent to the locus of σ as x generates $v=\text{const}$. *The congruence of lines* $(\rho\sigma)$ *is therefore central to* S *with respect to the given orthogonal net if and only if*

$$\frac{\partial^2}{\partial u \partial v} \log\left(\frac{M}{N}\right) = 0. \tag{3.6}$$

The net whose differential equation is

$$Edu^2 - Gdv^2 = 0 \tag{3.7}$$

is uniquely determined as that orthogonal net the tangents to whose curves separate the tangents to the curves of the given orthogonal

net harmonically. If the points ρ, σ defined by (3.4), (3.5) are defined using either one of the two families of curves of (3.7), it follows readily that *the congruence of such lines* $(\rho\sigma)$ *is central to* S *with respect to the given orthogonal net if, and only if, the net is isothermally orthogonal.*

As a final application of the notions in this paper we derive by geometrical methods all of the classical canonical lines except the projective normal. Suppose that the differential equations (1.1) are in the Fubini canonical form, that is, suppose $R=\beta\gamma$. Suppose that the line determined by the integral invariant points defined by (2.2) coincides with Bell's R_λ-associate [1] of the reciprocal of the projective normal for $\lambda = M/N$. It follows that

$$\frac{S}{M} = \gamma \frac{M}{N}, \qquad \frac{S}{N} = \beta \frac{N}{M}.$$

It follows that the curve C_λ is any one of the curves of Segre. *For these values of* M/N *the differential generating line determined by* (2.3) *is determined by*

$$\rho_h = x_u + \frac{h}{3}\phi x, \qquad \sigma_h = x_v + \frac{h}{3}\psi x,$$

wherein as usual

$$\phi = \frac{\partial}{\partial u}\log(\beta\gamma^2), \qquad \psi = \frac{\partial}{\partial v}\log(\beta^2\gamma).$$

That is *if the integral generating line is the* R_λ*-associate of the reciprocal of the projective normal, then the curve* C_λ *is a curve of Segre and the differential generating line is any one whatever of the classical canonical lines of the second kind.*

References

1. P. O. Bell, *A study of curved surfaces by means of certain associated ruled surfaces*, Trans. Amer. Math. Soc. vol. 46 (1939) pp. 389–409.

2. G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences*, Trans. Amer. Math. Soc. vol. 20 (1919) pp. 80–153.

3. V. G. Grove, *On canonical forms of differential equations*, Bull. Amer. Math. Soc. vol. 36 (1930) pp. 582–586.

4. ———, *Generalized canonical lines*, Duke Math. J. vol. 15 (1948) pp. 95–103.

5. E. P. Lane, *A treatise on projective differential geometry*, The University of Chicago Press, 1942.

Michigan State College

DEFORMATION THEORY OF SUBSPACES IN A RIEMANN SPACE

V. HLAVATÝ

Summary. This paper deals with the infinitesimal transformation (2,1) of a family (V_m) of subspaces V_m in a Riemann space. In §§1–4 the transformation (2,1) is applied on internal objects of a V_m, while in §§5 and 6 the characteristic mixed tensors $K^{\nu}_{a_r\cdots a_1}$ (cf. the equation (1,2; 4)) of a V_m are investigated with respect to (2,1). Finally, some applications of the theory are given in §§7 and 8. In particular the statement expressed by the equation (8,10)b is the generalization of the well known Levi-Civita result for $m=1$, while the statement expressed by the equation (8,10)c generalizes the classical result (for $m=1$, $n=2$) by Jacobi.

1. Preliminary.

(1) Let V_n be a n-dimensional Riemann space ($n \geqq 2$), referred to the real coördinate system ξ^{ν},[1] $g_{\lambda\mu}=g_{\mu\lambda}$ its metric tensor, $\Gamma^{\nu}_{\lambda\mu}=\Gamma^{\nu}_{\mu\lambda}$ the corresponding Christoffel symbols and ∇_{μ} the covariant derivative operator in V_n with respect to $\Gamma^{\nu}_{\lambda\mu}$. The curvature tensor of $\Gamma^{\nu}_{\lambda\mu}$ will be denoted by

$$(1,1;1) \qquad R^{\nu}_{\omega\mu\lambda} \equiv \partial_{\mu}\Gamma^{\nu}_{\lambda\omega} - \partial_{\omega}\Gamma^{\nu}_{\lambda\mu} + \Gamma^{\nu}_{\alpha\mu}\Gamma^{\alpha}_{\lambda\omega} - \Gamma^{\nu}_{\alpha\omega}\Gamma^{\alpha}_{\lambda\mu} \qquad \left(\partial_{\mu} \equiv \frac{\partial}{\partial\xi^{\mu}}\right).$$

(2) Let V_m be a m-dimensional subspace of V_n ($1 \leqq m < n$) referred to the real parameters[1] η^a and let

$$(1,2;1) \qquad \xi^{\nu} = \xi^{\nu}(\eta^1, \cdots, \eta^m)$$

be its parametric equations.[2] Throughout this paper we consider only the case where the matrix of the mixed tensor

$$(1,2;2) \qquad T^{\nu}_{a} \equiv \partial_a \xi^{\nu} \qquad \left(\partial_a \equiv \frac{\partial}{\partial\eta^a}\right)$$

is of rank m. Hence the metric tensor[3]

$$g_{ab} = g_{ba} = g_{\lambda\mu}T^{\lambda\mu}_{ab}$$

Received by the editors May 31, 1949 and, in revised form, July 11, 1949.

[1] Greek (Latin) indices run from $\dot{1}$ to $\dot{n}$ (from 1 to m).

[2] Any function considered in this paper is understood to be a real and continuous one, as well as its derivatives which appear in the discussion.

[3] Here and later on we put $T^{\lambda\cdots\nu}_{a_1\cdots a_\nu} \equiv T^{\lambda}_{a_1\cdots}T^{\nu}_{a_\nu}$, $\nu=2, 3, \cdots$.

is of rank m and consequently the corresponding Christoffel symbols $\Gamma^c_{ab}=\Gamma^c_{ba}$ exist. The curvature tensor of V_m will be denoted by

$$P^a_{dcb} \equiv \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^a_{hc}\Gamma^h_{bd} - \Gamma^a_{hd}\Gamma^h_{bc}.$$

Considering the connections $\Gamma^\nu_{\lambda\mu}$ and Γ^a_{bc} we may introduce three different kinds of covariant derivatives, namely

$$(1,2;3a) \qquad \nabla_\mu V^{\nu\cdots}_{\lambda\cdots} \equiv \partial_\mu V^{\nu\cdots}_{\lambda\cdots} + \Gamma^\nu_{\alpha\mu}V^{\alpha\cdots}_{\lambda\cdots} + \cdots - \Gamma^\alpha_{\lambda\mu}V^{\nu\cdots}_{\alpha\cdots} - \cdots,$$

for a tensor field defined over V_n,

$$(1,2;3b) \nabla_a V^{\nu\cdots}_{\lambda\cdots} \equiv \partial_a V^{\nu\cdots}_{\lambda\cdots} + T^\mu_a\{\Gamma^\nu_{\alpha\mu}V^{\alpha\cdots}_{\lambda\cdots} + \cdots - \Gamma^\alpha_{\lambda\mu}V^{\nu\cdots}_{\alpha\cdots} - \cdots\},$$

for a tensor field defined over V_m and

$$(1,2;3c) \qquad D_b V^{\nu\cdots}_{a\cdots} \equiv \partial_b V^{\nu\cdots}_{a\cdots} + \Gamma^\nu_{\lambda\mu}T^\mu_b V^{\lambda\cdots}_{a\cdots} + \cdots - \Gamma^c_{ab}V^{\nu\cdots}_{c\cdots} - \cdots,$$

for a mixed tensor field defined over V_m. The embedding theory of a V_m in a V_n may be described by means of the mixed tensors T^ν_a and

$$(1,2;4) \qquad K^\nu_{a_r\cdots a_1} \equiv D_{a_r}\ldots D_{a_2}T^\nu_{a_1}, \qquad r=2,\cdots,N,$$

where N is the number of osculating spaces of V_m.[4]

Incidentally $K^\nu_{ab}=K^\nu_{ba}$ lies with its index ν in the first normal space of V_m.

(3) A family (V_m) of a set of V_m's is defined as a set of V_m's such that through any generic point of the V_n there is only one element V_m of (V_m). Each element V_m of the family (V_m) is referred to the parameter system η^a and we keep the parameter transformation $\eta^{a'}\leftrightarrow\eta^a$ independent of the coördinate system ξ^ν. Hence

$$(1,3;1) \qquad \partial_\mu\frac{\partial\eta^{a'}}{\partial\eta^a}=0 \qquad \partial_\mu\frac{\partial\eta^a}{\partial\eta^{a'}}=0.$$

On the other hand, an object Ω defined over (V_m) is defined also over V_n and consequently may be thought of as expressed either by means of the η^a or by means of the ξ^ν. This is in particular true for T^ν_a, g_{ab}, Γ^c_{ab}. Whenever we apply on such an object the operator ∂_μ it is understood that we consider it expressed by means of the ξ^ν. Taking in account (1,3;1) we see that[5] $\partial_\mu g_{ab}$, $\partial_\mu\Gamma^c_{ab}$, $\nabla_\omega T^\nu_a$ are

[4] Cf. V. Hlavatý: *Embedding theory of a W_m in a W_n* (to be published in Actualités Scientifiques et Industrielles) where the case of a Riemann space is included. Cf. also J. A. Schouten and E. R. van Kampen, *Über die Krümmung einer V_m in V_n; eine Revision der Krümmungstheorie* (Math. Ann. vol. 105, p. 144–159).

[5] $\nabla_\omega T^\nu_a \equiv \partial_\omega T^\nu_a + \Gamma^\nu_{\lambda\omega}T^\lambda_a$ according to (1,2;3a), while (cf. (1,2;3b)) $\nabla_b T^\nu_a \equiv \partial_b T^\nu_a + \Gamma^\nu_{\lambda\omega}T^{\lambda\omega}_{ab}$.

tensors (resp. vectors) not only with respect to the coördinate transformation but also with respect to the parameter transformation. Let P be a generic point of V_n and let V_m be the element of (V_m) containing P. If a tensor field T defined over V_n has the property that $T(P)$ is in the tangential space (or in some of the normal spaces) of the V_m already mentioned, then we say that the field T is tangent (normal) to (V_m).

2. **Fundamental definitions.** Let $V^\nu = V^\nu(\xi)$ be the components of a contravariant vector field given over V_n and let

$$^*\xi^\nu = \xi^\nu + \epsilon V^\nu \qquad (\epsilon \to 0 \text{ is a constant}) \tag{2,1}$$

be the infinitesimal transformation of a Lie group with the generator $V^\lambda \partial_\lambda$. If a generic point $P(\xi)$ describes a subspace V_m of the family (V_m) then the point $^*P \equiv P(^*\xi)$ describes a subspace $^*V_m$. We shall assume throughout this paper that the set of all $^*V_m$ constructed in this way is a family, which we denote by $(^*V_m)$. In the following definition $X = X(P)$ denotes a tensor field (a set of tensor fields) defined either over (V_m) or over V_n and $F[X] \equiv F(\xi)$ is a function of X.

DEFINITION (2,1). *X' is the value of X in $^*P$, $X' \equiv X(^*P)$, $^\circ X$ is the tensor at $^*P$ which one gets by parallel displacement in V_n of $X(P)$ from P to $^*P$. If X is defined over (V_m), then $^*X$ is the tensor field (a set of tensor fields) defined over $(^*V_m)$ in the same way as X is defined over (V_m). If X is defined over V_n, then $^*X \equiv X'$. Furthermore*

$$^*F \equiv F[^*X], \qquad ^\circ F \equiv F[^\circ X], \qquad F' \equiv F(^*\xi) \tag{2,2a}$$

in the first case and

$$^*F \equiv F' \equiv F(^*\xi), \qquad ^\circ F \equiv F[^\circ X] \tag{2,2b}$$

in the second case.

We use these symbols in the following definition of the symbols τ, ω, and Δ:

DEFINITION (2,2). *The operators τ, ω, Δ are defined by the following equations*

$$\begin{aligned} &\text{(a)} \quad \tau F \equiv \lim_{\epsilon \to 0} \frac{^*F - {}^\circ F}{\epsilon} && (\textit{translation of } F), \\ &\text{(b)} \quad \omega F \equiv \lim_{\epsilon \to 0} \frac{F' - {}^\circ F}{\epsilon} && (\textit{variation of } F), \\ &\text{(c)} \quad \Delta F \equiv (\tau - \omega)F && (\textit{deformation of } F). \end{aligned} \tag{2,3}$$

In this paper we shall investigate the application of these operators

on the objects of (V_m).

3. The fields T^r_a and g_{ab}.

THEOREM (3,1). *We have*

$$(3,1)\qquad (a)\ \tau T^r_a = \nabla_a V^r, \qquad (b)\ \omega T^r_a = V^\mu \nabla_\mu T^r_a,$$
$$(c)\ \Delta T^r_a = L^r_a,$$

where

$$(3,2)\qquad L^r_a \equiv \nabla_a V^r - V^\mu \nabla_\mu T^r_a = \partial_a V^r - V^\mu \partial_\mu T^r_a.$$

PROOF. Let $X \equiv F[X] \equiv T^r_a$. Then

$$*F \equiv \frac{\partial^* \xi^r}{\partial \eta^a} = T^r_a + \epsilon \partial_a V^r, \qquad F' \equiv T^r_a(^*\xi) = T^r_a + \epsilon V^\mu \partial_\mu T^r_a + \cdots,^6$$
$$^\circ F \equiv {}^\circ T^r_a = T^r_a - \epsilon V^\mu \Gamma^r_{\lambda\mu} T^\lambda_a + \cdots$$

and consequently

$$(3,3)\qquad (a)\ *F - {}^\circ F = T^r_a + \epsilon(\partial_a V^r + \Gamma^r_{\lambda\mu} T^\mu_a V^\lambda) - T^r_a + \cdots,$$
$$(b)\ F' - {}^\circ F = T^r_a + \epsilon V^\mu(\partial_\mu T^r_a + \Gamma^r_{\lambda\mu} T^\lambda_a) - T^r_a + \cdots.$$

From (3,3)ab we have (3,1)ab and these equations together with (3,2) lead to (3,1)c.

THEOREM (3,2). *We have*

$$(3,4)\qquad (a)\ \tau g_{ab} = 2W_{ab},^7 \qquad (b)\ \omega g_{ab} = G_{ab},$$
$$(c)\ \Delta g_{ab} = 2 g_{\lambda\mu} L^\lambda_{(a} T^\mu_{b)}$$

where

$$(3,5)\qquad W_{ab} \equiv T^{\lambda\mu}_{(ab)} \nabla_\lambda V_\mu, \qquad G_{ab} \equiv V^\mu \partial_\mu g_{ab}.$$

PROOF. Put $X \equiv g_{\lambda\mu}$. Then

$$(3,6)\qquad (a)\ *X \equiv X' = {}^*g_{\lambda\mu} = g_{\lambda\mu} + \epsilon V^\omega \partial_\omega g_{\lambda\mu} + \cdots,$$
$$(b)\ {}^\circ X \qquad = {}^\circ g_{\lambda\mu} = g_{\lambda\mu} + \epsilon V^\omega(\Gamma^r_{\lambda\omega} g_{r\mu} + \Gamma^r_{\mu\omega} g_{\lambda r}) + \cdots = {}^*g_{\lambda\mu}.$$

On the other hand, if we denote by X the set of tensor fields $g_{\lambda\mu}$, T^r_a we have for $F[X] \equiv g_{\lambda\mu} T^\lambda_a T^\mu_b = g_{ab}$ by virtue of (3,6)

[6] Throughout this paper the dots denote the coefficient of ϵ^2.

[7] Substantially equivalent formulas to (3,1)a and (3,4)a may be found also in the papers in the bibliography, which deal with the theory of deformation of subspaces from a different point of view.

$$
\begin{aligned}
\text{(a)}\quad {}^{*}F \equiv {}^{*}g_{ab} &= {}^{*}g_{\lambda\mu}{}^{*}T_a^{\lambda}{}^{*}T_b^{\mu} = (g_{\lambda\mu} + \epsilon V^{\omega}\partial_{\omega}g_{\lambda\mu} + \cdots)\\
&\quad \cdot (T_a^{\lambda} + \epsilon\partial_a V^{\lambda} + \cdots)(T_b^{\mu} + \epsilon\partial_b V^{\mu} + \cdots)\\
&= g_{ab} + \epsilon[V^{\omega}T_a^{\lambda}T_b^{\mu}\partial_{\omega}g_{\lambda\mu} + 2g_{\lambda\mu}T_{(b}^{\mu}\partial_{a)}V^{\lambda}] + \cdots\\
&= g_{ab} + 2\epsilon T_{(a}^{\lambda}T_{b)}^{\mu}\nabla_{\lambda}V_{\mu} + \cdots,\\
\text{(3,7)}\quad \text{(b)}\quad F' \equiv g'_{ab} &= g'_{\lambda\mu}T_a^{\lambda'}T_b^{\mu'} = (g_{\lambda\mu} + \epsilon V^{\omega}\partial_{\omega}g_{\lambda\mu} + \cdots)\\
&\quad \cdot (T_a^{\lambda} + \epsilon V^{\rho}\partial_{\rho}T_a^{\lambda} + \cdots)(T_b^{\mu} + \epsilon V^{\sigma}\partial_{\sigma}T_b^{\mu} + \cdots)\\
&= g_{ab} + \epsilon[V^{\omega}T_a^{\lambda}T_b^{\mu}\partial_{\omega}g_{\lambda\mu} + 2g_{\lambda\mu}V^{\omega}(\partial_{\omega}T_{(a}^{\lambda})T_{b)}^{\mu}] + \cdots\\
&= g_{ab} + \epsilon V^{\omega}\partial_{\omega}g_{ab} + \cdots,\\
\text{(c)}\quad {}^{\circ}F \equiv {}^{\circ}g_{ab} &= {}^{\circ}g_{\lambda\mu}{}^{\circ}T_a^{\lambda}{}^{\circ}T_b^{\mu} = [g_{\lambda\mu} + \epsilon V^{\omega}(\Gamma_{\lambda\omega}^{\nu}g_{\nu\mu} + \Gamma_{\mu\omega}^{\nu}g_{\lambda\nu}) + \cdots]\\
&\quad \cdot (T_a^{\lambda} - \epsilon V^{\rho}\Gamma_{a\rho}^{\lambda}T_a^{\alpha} + \cdots)(T_b^{\mu} - \epsilon V^{\sigma}\Gamma_{\beta\sigma}^{\mu}T_b^{\beta} + \cdots)\\
&= g_{ab} + \epsilon[T_a^{\lambda}T_b^{\mu}V^{\omega}(\Gamma_{\lambda\omega}^{\nu}g_{\nu\mu} + \Gamma_{\mu\omega}^{\nu}g_{\lambda\nu}) - 2g_{\lambda\mu}T_{(b}^{\mu}T_{a)}^{\alpha}V^{\omega}\Gamma_{a\omega}^{\lambda}]\\
&\quad + \cdots = g_{ab} + \cdots.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\text{(d)}\quad {}^{*}F - {}^{\circ}F \equiv {}^{*}g_{ab} - {}^{\circ}g_{ab} = 2\epsilon T_{(b}^{\mu}T_{a)}^{\lambda}\nabla_{\lambda}V_{\mu} + \cdots,\\
\text{(3,7)}\quad &\text{(e)}\quad F' - {}^{\circ}F = g'_{ab} - {}^{\circ}g_{ab} = \epsilon V^{\omega}\partial_{\omega}g_{ab} + \cdots.
\end{aligned}
$$

These equations lead to (3,4)ab and from these we have (3,4)c.

DEFINITION (3,1). *The transformation* (2,1) *is called a rigid translation if* $\tau g_{ab} = 0$.

THEOREM (3,3a). *Let* V^{ν} *be a tangential vector field to* (V_m). *A necessary and sufficient condition that* (2,1) *be a rigid translation is the* (*Killing*) *equation*

$$
D_{(a}V_{b)} = 0. \tag{3,8}
$$

PROOF. In our case we have[8] $V_a = V_{\lambda}T_a^{\lambda}$, $V_{\lambda} = V_c T_{\lambda}^{c}$ and consequently (by virtue of $\Gamma_{ab}^{c} = T_{\lambda}^{c}\nabla_b T_a^{\lambda}$)

$$
\begin{aligned}
D_b V_a &= \partial_b V_{\lambda}T_a^{\lambda} - T_{\lambda}^{c}(\nabla_b T_a^{\lambda})V_c = (\partial_b V_{\lambda})T_a^{\lambda} + V_{\lambda}\partial_b T_a^{\lambda} - (\nabla_b T_a^{\lambda})V_{\lambda}\\
&= T_a^{\nu}(\partial_b V_{\nu} - \Gamma_{\mu\nu}^{\lambda}T_b V_{\lambda}^{\mu}) = T_b T_a^{\mu}\nabla_{\mu}^{\lambda}V_{\lambda}.
\end{aligned} \tag{3,9}
$$

The proof follows from (3,9), (3,5), and (3,4)a.

THEOREM (3,3b). *Let* V^{ν} *be a normal vector field to* (V_m). *A necessary and sufficient condition that* (2,1) *be a rigid translation is*: V^{ν} *is in the* x*th normal space of* V_m, $x = 2, 3, \cdots$.

[8] $T_{\lambda}^{c} \equiv T_a^{\nu}g^{ac}g_{\nu\lambda}$.

PROOF. In our case we have $V_\lambda T_a^\lambda = 0$ and consequently

$$T_a^\lambda T_b^\mu \nabla_\lambda V_\mu = - T_a^\lambda V_\mu \nabla_\lambda T_b^\mu = - (D_a T_b^\mu) V_\mu = - K_{ab}^\mu V_\mu. \tag{3,10}$$

The tensor K_{ab}^μ lies with μ in the first normal space of (V_m). By virtue of this fact, we get the theorem (3,3b) from (3,4)a and (3,5).

REMARK. The theorem (3,3b) is the generalization of the well known fact that the "deformation" of the arc of a curve in a three-dimensional Euclidean space along its binormal is equal to zero.

4. The field Γ_{ab}^c.

THEOREM (4,1). *We have*

$$\text{(a)}\quad {}_\tau\Gamma_{ab}^c = W_{ab}^c \equiv W_{abd} g^{dc}, \qquad \text{(b)}\quad {}_\omega\Gamma_{ab}^c = G_{ab}^c, \qquad \text{(c)}\quad \Delta\Gamma_{ab}^c = Q_{ab}^c \tag{4,1}$$

where

$$\begin{aligned} &\text{(a)}\quad W_{abd} \equiv D_a W_{db} + D_b W_{da} - D_d W_{ba}, \\ &\text{(b)}\quad G_{ab}^c = V^\omega \partial_\omega \Gamma_{ab}^c, \\ &\text{(c)}\quad Q_{ab}^c \equiv W_{abd} g^{dc} - G_{ab}^c. \end{aligned} \tag{4,2}$$

PROOF. First of all we have

$$\partial_a W_{db} + \partial_b W_{da} - \partial_d W_{ab} = W_{abd} + 2\Gamma_{ba}^e W_{de}. \tag{4,3}$$

Furthermore, if X is the set of tensors g_{ab}, g^{cd}, we get for $F[X] \equiv \Gamma_{ab}^c$ by virtue of (3,7)d and (4,3)

$$\begin{aligned} {}^*F \equiv {}^*\Gamma_{ab}^c &= 1/2\, {}^*g^{cd}(\partial_a {}^*g_{bd} + \partial_b {}^*g_{ad} - \partial_d {}^*g_{ab}) \\ &= 1/2[{}^\circ g^{cd} - 2\epsilon W^{cd} + \cdots][\partial_a({}^\circ g_{bd} + 2\epsilon W_{bd} + \cdots) \\ &\quad + \partial_b({}^\circ g_{ad} + 2\epsilon W_{ad} + \cdots) \\ &\quad - \partial_d({}^\circ g_{ab} + 2\epsilon W_{ab} + \cdots)] \\ &= {}^\circ\Gamma_{ba}^c + \epsilon\{{}^\circ g^{cd}[W_{abd} + 2\Gamma_{ba}^e W_{de}] \\ &\quad - 2W^{cd}\, {}^\circ\Gamma_{ba}^e {}^\circ g_{ed}\} + \cdots. \end{aligned} \tag{4,4}$$

On the other hand, we see from (3,7)c that

$${}^\circ\Gamma_{ba}^c = \Gamma_{ba}^c + \cdots. \tag{4,5}$$

From (4,5), (4,4) we have (4,1)a. The equation (4,1)b follows from

$$F' \equiv \Gamma_{ab}^{c\prime} = \Gamma_{ab}^c + \epsilon V^\omega \partial_\omega \Gamma_{ab}^c + \cdots \tag{4,6}$$

and (4,5). The equation (4,1)c is obvious.

Later on we shall need also the coefficients

$$(4,7) \qquad {}'\Gamma^{e}_{ab} = 1/2 g^{ed\prime}(\partial_a g'_{bd} + \partial_b g'_{ad} - \partial_d g'_{ab})$$

which are related to $\Gamma^{e\prime}_{ab}$ by a relation introduced in the following lemma:

LEMMA (4,1). *We have*

$$(4,8) \qquad {}'\Gamma^{e}_{ab} - \Gamma^{e\prime}_{ab} = \frac{\epsilon}{2} g^{ed}[L^{\omega}_{a}\partial_\omega g_{db} + L^{\omega}_{b}\partial_\omega g_{da} - L^{\omega}_{d}\partial_\omega g_{ab}] + \cdots$$

PROOF. We get from (4,7) and (3,7)b

$$\begin{aligned}
{}'\Gamma^{e}_{ab} &= 1/2(g^{ed} + \epsilon V^{\omega}\partial_\omega g^{ed} + \cdots)[\partial_a(g_{bd} + \epsilon V^{\rho}\partial_\rho g_{bd} + \cdots) \\
&\quad + \partial_b(g_{ad} + \epsilon V^{\rho}\partial_\rho g_{ad} + \cdots) \\
&\quad - \partial_d(g_{ab} + \epsilon V^{\rho}\partial_\rho g_{ab} + \cdots)] \\
&= \Gamma^{e}_{ab} + \frac{\epsilon}{2}\{2\Gamma^{\lambda}_{ab}g_{\lambda d}V^{\rho}\partial_\rho g^{ed} + g^{ed}[(\partial_a V^{\rho})\partial_\rho g_{bd} \\
&\quad + (\partial_b V^{\rho})\partial_\rho g_{ad} - (\partial_e V^{\rho})\partial_\rho g_{ab} \\
&\quad + g^{ed}V^{\rho}(\partial_a\partial_\rho g_{bd} + \partial_b\partial_\rho g_{ad} - \partial_d\partial_\rho g_{ab})]\} + \cdots .
\end{aligned} \tag{4,9}$$

On the other hand

$$\begin{aligned}
\Gamma^{e\prime}_{ab} &= \Gamma^{e}_{ab} + \epsilon V^{\rho}\partial_\rho\Gamma^{e}_{ab} + \cdots = \Gamma^{e}_{ab} + \epsilon[V^{\rho}(\partial_\rho g^{ed})\Gamma^{\lambda}_{ab}g_{\lambda d} \\
&\quad + 1/2 g^{ed}V^{\rho}(\partial_\rho\partial_a g_{bd} + \partial_\rho\partial_b g_{ad} - \partial_\rho\partial_d g_{ab})] + \cdots ,
\end{aligned} \tag{4,10}$$

$$\begin{aligned}
(\partial_a\partial_\rho - \partial_\rho\partial_a)g_{bd} &= (T^{\lambda}_{a}\partial_\lambda\partial_\rho - (\partial_\rho T^{\lambda}_{a})\partial_\lambda - T^{\lambda}_{a}\partial_\rho\partial_\lambda)g_{bd} \\
&= -(\partial_\rho T^{\lambda}_{a})(\partial_\lambda g_{bd}).
\end{aligned} \tag{4,11}$$

Comparing (4,9), (4,10), and (4,11), we obtain

$$\begin{aligned}
{}'\Gamma^{e}_{ab} - \Gamma^{e\prime}_{ab} &= \frac{\epsilon}{2} g^{ed}[(\partial_a V^{\rho} - V^{\lambda}\partial_\lambda T^{\rho}_{a})\partial_\rho g_{bd} + (\partial_b V^{\rho} - V^{\lambda}\partial_\lambda T^{\rho}_{b})\partial_\rho g_{ad} \\
&\quad - (\partial_d V^{\rho} - V^{\lambda}\partial_\lambda T^{\rho}_{d})\partial_\rho g_{ab} + \cdots]
\end{aligned}$$

and this equation gives us at once (4,8).

5. **The field $K^{\nu}_{\lambda\mu}$.**[9] The first theorem concerns the translation of the

[9] Throughout §§5 and 6 we shall sometimes make use of a special coördinate (parameter) system and shall symbolize the statement: "in the special coördinate (parameter) system $\Gamma^{\nu}_{\lambda\mu}=0$ ($\Gamma^{e}_{ab}=0$) at P" by using the symbol $\doteq$, $\Gamma^{\nu}_{\lambda\mu}\doteq 0$ ($\Gamma^{e}_{ab}\doteq 0$). An equation with $\doteq$ has to be understood as valid in a special coördinate (parameter) system at P where both last equations hold. On the other hand, the sign $=$ does not impose any restriction whatsoever on the choice of the coördinate (parameter) system.

tensor

$$K^{\nu}_{ba} = K^{\nu}_{ab} \equiv D_b T^{\nu}_a. \tag{5,1}$$

THEOREM (5,1). *The following equation holds*:

$$\tau K^{\nu}_{ba} = D_b D_a V^{\nu} - V^{\omega} T^{\lambda\mu}_{ba} R^{\nu}_{\omega\mu\lambda} - W^{c}_{ba} T^{\nu}_{c}. \tag{5,2}$$

PROOF. First of all we have according to (2,2)b for $X \equiv g_{\lambda\mu}$, $g^{\lambda\nu}$, $F[X] \equiv F(\xi) = \Gamma^{\nu}_{\lambda\mu}$

$$*\Gamma^{\nu}_{\lambda\mu} = \Gamma^{\nu'}_{\lambda\mu} \doteq \epsilon V^{\omega} \partial_{\omega} \Gamma^{\nu}_{\lambda\mu} + \cdots \tag{5,3)a}$$

and moreover by virtue of (4,4) and (4,5)

$$*\Gamma^{c}_{ba} \doteq \epsilon W^{c}_{ba} + \cdots . \tag{5,3)b}$$

Hence

$$\begin{aligned} *K^{\nu}_{ba} = *D_b *T^{\nu}_a &\doteq \partial_b (T^{\nu}_a + \epsilon \partial_a V^{\nu} + \cdots) \\ &+ \epsilon V^{\rho} (\partial_{\rho} \Gamma^{\nu}_{\lambda\mu})(T^{\mu}_b + \epsilon \partial_b V^{\mu} + \cdots)(T^{\lambda}_a + \epsilon \partial_a V^{\lambda} + \cdots) \\ &- \epsilon W^{c}_{ba} (T^{\nu}_c + \epsilon \partial_c V^{\nu} + \cdots) \end{aligned} \tag{5,4)a}$$

and consequently

$$\begin{aligned} *K^{\nu}_{ba} &\doteq \partial_b T^{\nu}_a + \epsilon \{ \partial_b \partial_a V^{\nu} + V^{\rho} (\partial_{\rho} \Gamma^{\nu}_{\lambda\mu}) T^{\mu\lambda}_{ba} - W^{c}_{ba} T^{\nu}_c \} + \cdots \\ &\doteq D_b T^{\nu}_a + \epsilon \{ D_b D_a V^{\nu} - V^{\omega} [\partial_{\mu} \Gamma^{\nu}_{\omega\lambda} - \partial_{\omega} \Gamma^{\nu}_{\mu\lambda}] T^{\mu\lambda}_{ba} - W^{c}_{ba} T^{\nu}_c \} \\ &\quad + \cdots \\ &\doteq D_b T^{\nu}_a + \epsilon \{ D_b D_a V^{\nu} - V^{\omega} T^{\mu\lambda}_{ba} R^{\nu}_{\omega\mu\lambda} - W^{c}_{ba} T^{\nu}_c \} + \cdots . \end{aligned} \tag{5,4)b}$$

On the other hand

$$^{\circ}K^{\nu}_{ba} \doteq D_b T^{\nu}_a + \cdots = K^{\nu}_{ba} + \cdots . \tag{5,4)c}$$

The equations (5,4)b and (5,4)c lead to (5,2) with the $\doteq$ sign. This equation, being a tensor equation valid for a special coördinate (and parameter) system at a generic point P, holds for all coördinate (and parameter) systems at a generic point.

The second theorem concerns the variation of K^{ν}_{ab}:

THEOREM (5,2). *The following equation holds*:

$$\omega K^{\nu}_{ba} = V^{\omega} [D_b \nabla_{\omega} T^{\nu}_a - T^{\mu\lambda}_{ba} R^{\nu}_{\omega\mu\lambda} + (\nabla_{\mu} T^{\nu}_a)(\nabla_{\omega} T^{\mu}_b)] - G^{c}_{ba} T^{\nu}_c. \tag{5,5}$$

PROOF. First of all we have

$$\partial_\omega\partial_b T^\nu_a - \partial_b\partial_\omega T^\nu_a = \partial_\omega T^\mu_b \partial_\mu T^\nu_a - T^\mu_b \partial_\mu\partial_\omega T^\nu_a = (\partial_\omega T^\mu_b)(\partial_\mu T^\nu_a)$$
$$\doteq (\nabla_\omega T^\mu_b)(\nabla_\mu T^\nu_a),$$
$$D_b\nabla_\omega T^\nu_a \doteq \partial_b\partial_\omega T^\nu_a + T^{\mu\lambda}_{ba}(\partial_\mu\Gamma^\nu_{\lambda\omega}).$$

Consequently

$$\begin{aligned} K''^\nu_{ba} &\equiv (D_b T^\nu_a)' = K^\nu_{ba} + \epsilon V^\omega \partial_\omega K^\nu_{ba} + \cdots \\ &\doteq K^\nu_{ba} + \epsilon V^\omega [\partial_\omega\partial_b T^\nu_a + (\partial_\omega\Gamma^\nu_{\lambda\mu})T^{\mu\lambda}_{ba} - (\partial_\omega\Gamma^c_{ab})T^\nu_c] + \cdots \\ &= K^\nu_{ba} + \epsilon V^\omega \{ [D_b\nabla_\omega T^\varrho_a + T^{\mu\lambda}_{ba}(\partial_\omega\Gamma^\nu_{\lambda\mu} - \partial_\mu\Gamma^\nu_{\lambda\omega}) \\ &\quad + (\nabla_\omega T^\mu_b)(\nabla_\mu T^\nu_a)] - G^c_{ba} T^\nu_c \}. \end{aligned} \tag{5,6}$$

The proof follows from (5,6) and (5,4)c.

THEOREM (5,3). *Put*

$$L^\nu_{ba} \equiv D_b L^\nu_a + L^\mu_b \nabla_\mu T^\nu_a - Q^c_{ba} T^\nu_c. \tag{5,7)a}$$

Then

$$\begin{aligned} L^\nu_{ba} = {} & D_b L^\nu_a - V^\omega T^{\mu\lambda}_{ba} R^\nu_{\omega\mu\lambda} - Q^c_{ba} T_c \\ & + (T^\mu_b \nabla_\mu V^\omega \nabla_\omega - V^\omega \nabla_\omega T^\mu_b \nabla_\mu) T^\nu_a \end{aligned} \tag{5,7)b}$$

and

$$\Delta K^\nu_{ba} = L^\nu_{ba}. \tag{5,8}$$

PROOF. We have from (5,2) and (5,5)

$$\begin{aligned} \Delta K^\nu_{ba} = (\tau - \omega) K^\nu_{ba} = {} & D_b D_a V^\nu - V^\omega D_b \nabla_\omega T^\nu_a \\ & - V^\omega (\nabla_\omega T^\mu_b) \nabla_\mu T^\varkappa_a - Q^c_{ba} T^\nu_c. \end{aligned} \tag{5,9}$$

Since

$$\begin{aligned} D_b L^\nu_a &= D_b [D_a V^\nu - V^\varrho \nabla_\varrho T^\nu_a] \\ &= D_b D_a V^\nu - (D_b V^\varrho)\nabla_\varrho T^\nu_a - V^\omega D_b \nabla_\omega T^\nu_a, \end{aligned}$$

the equation (5,9) reduces to

$$\Delta K^\nu_{ba} = D_b L^\nu_a + (\nabla_\mu T^\nu_a)(\nabla_b V^\mu - V^\varrho \nabla_\varrho T^\mu_b) - Q^c_{ba} T^\nu_c$$

which proves (5,7)a and (5,8). The equation (5,7)b follows from (5,7)a and

$$\begin{aligned} (T^\mu_b \nabla_\mu V^\omega \nabla_\omega - V^\omega \nabla_\omega T^\mu_b \nabla_\mu) T^\nu_a &= [L^\mu_b \nabla_\mu + V^\omega T^\mu_b (\nabla_\mu\nabla_\omega - \nabla_\omega\nabla_\mu)] T^\nu_a \\ &= L^\mu_b \nabla_\mu T^\nu_a - V^\omega T^\mu_b T^\lambda_a R^\nu_{\mu\omega\lambda} \end{aligned} \tag{5,10}$$

6. **The field $K^{\nu}_{a_r\cdots a_1}$.** The last theorem may easily be generalized:

THEOREM (6,1). *Put*

$$L^{\nu}_{a_r\cdots a_1} \equiv D_{a_r}L^{\nu}_{a_{r-1}\cdots a_1} + L^{\mu}_{a_r}\nabla_{\mu}K^{\nu}_{a_{r-1}\cdots a_1} - \sum_1^{r-1}{}_s Q^{c}_{a_s a_r}K^{\nu}_{a_{r-1}\cdots a_{s+1}ca_{s-1}\cdots a_1}. \tag{6,1)a}$$

Then

$$L^{\nu}_{a_r\cdots a_1} = D_{a_r}L^{\nu}_{a_{r-1}\cdots a_1} + (T^{\mu}_{a_r}\nabla_{\mu}V^{\omega}\nabla_{\omega} - V^{\omega}\nabla_{\omega}T^{\mu}_{a_r}\nabla_{\mu})K^{\nu}_{a_{r-1}\cdots a_1} - V^{\omega}T^{\mu}_{a_r}R^{\nu}_{\omega\mu\lambda}K^{\lambda}_{a_{r-1}\cdots a_1} - \sum_1^{r-1}{}_s Q^{c}_{a_s a_r}K^{\nu}_{a_{r-1}\cdots a_{s+1}ca_{s-1}\cdots a_1} \tag{6,1)b}$$

and

$$\Delta K^{\nu}_{a_r\cdots a_1} = L^{\nu}_{a_r\cdots a_1} \qquad (r = 2, 3, \cdots). \tag{6,2}$$

PROOF. (The theorem has already been proved for $r=2$.) Let us assume that we have proved it for some $r\geqq 2$. Then we have from (6,2)

$$\begin{aligned} {}^*K^{\nu}_{a_r\cdots a_1} &= K'^{\nu}_{a_r\cdots a_1} + \epsilon L^{\nu}_{a_r\cdots a_1} + \cdots \\ &= K^{\nu}_{a_r\cdots a_1} + \epsilon[L^{\nu}_{a_r\cdots a_1} + V^{\omega}\partial_{\omega}K^{\nu}_{a_r\cdots a_1}] + \cdots . \end{aligned} \tag{6,3}$$

On the other hand

$${}^*D_{a_{r+1}}{}^*K^{\nu}_{a_r\cdots a_1} \doteq \partial_{a_{r+1}}{}^*K^{\nu}_{a_r\cdots a_1} + \epsilon\Big[V^{\omega}(\partial_{\omega}\Gamma^{\nu}_{\lambda\mu})T^{\mu}_{a_{r+1}}K^{\lambda}_{a_r\cdots a_1} - \sum_1^{r}{}_s W^{c}_{a_s a_{r+1}}K^{\nu}_{a_r\cdots a_{s+1}ca_{s-1}\cdots a_1}\Big] + \cdots$$

and consequently by virtue of (6,3)

$$\begin{aligned} {}^*K^{\nu}_{a_{r+1}\cdots a_1} \doteq \partial_{a_{r+1}}K^{\nu}_{a_r\cdots a_1} + \epsilon\Big\{&\partial_{a_{r+1}}L^{\nu}_{a_r\cdots a_1} \\ &+ (\partial_{a_{r+1}}V^{\omega})\partial_{\omega}K^{\nu}_{a_r\cdots a_1} + V^{\omega}\partial_{a_{r+1}}\partial_{\omega}K^{\nu}_{a_r\cdots a_1} \\ &+ V^{\omega}(\partial_{\omega}\Gamma^{\nu}_{\lambda\mu})T^{\mu}_{a_{r+1}}K^{\lambda}_{a_r\cdots a_1} \\ &- \sum_1^{r}{}_s W^{c}_{a_s a_{r+1}}K^{\nu}_{a_r\cdots a_{s+1}ca_{s-1}\cdots a_1}\Big\} + \cdots \end{aligned} \tag{6,4}$$

$$\doteq D_{a_{r+1}}K^{\nu}_{a_r\cdots a_1} + \epsilon\Big\{D_{a_{r+1}}L^{\nu}_{a_r\cdots a_1}$$

$$+ (\nabla_{a_{r+1}}V^{\rho})\nabla_{\rho}K^{\nu}_{a_r\cdots a_1}$$

$$- \sum_{1}^{r}{}_{s} W^{\sigma}_{a_s a_{r+1}} K^{\nu}_{a_r\cdots a_{s+1}\sigma a_{s-1}\cdots a_1} \tag{6,4}$$

$$+ V^{\omega}[(\partial_{\omega}\Gamma^{\nu}_{\lambda\mu})T^{\mu}_{a_{r+1}}K^{\lambda}_{a_r\cdots a_1} + \partial_{a_{r+1}}\partial_{\omega}K^{\nu}_{a_r\cdots a_1}]\Big\}$$

$$+ \cdots .$$

The tensor $K^{\nu}_{a_{r+1}\ldots a_1}$ satisfies the following relation

$$\begin{aligned} K'^{\nu}_{a_{r+}\cdots a} &= K^{\nu}_{a_{r+1}\cdots a_1} + \epsilon V^{\omega}\partial_{\omega}K^{\nu}_{a_{r+1}\cdots a_1} + \cdots \\ &= K^{\nu}_{a_{r+1}\cdots a_1} + \epsilon V^{\omega}\partial_{\omega}D_{a_{r+1}}K^{\nu}_{a_r\cdots a_1} + \cdots \\ &\doteq K^{\nu}_{a_{r+1}\cdots a_1} + \epsilon V^{\omega}\Big[\partial_{\omega}\partial_{a_{r+1}}K^{\nu}_{a_r\cdots a_1} \end{aligned} \tag{6,5}$$

$$+ (\partial_{\omega}\Gamma^{\nu}_{\lambda\mu})T^{\mu}_{a_{r+1}}K^{\lambda}_{a_r\cdots a_1}$$

$$- \sum_{1}^{r}{}_{s}(\partial_{\omega}\Gamma^{\sigma}_{a_s a_{r+1}})K^{\nu}_{a_r\cdots a_{s+1}\sigma a_{s-1}\cdots a_1}\Big] + \cdots .$$

Subtracting (6,5) from (6,4) we have

$$*K^{\nu}_{a_{r+1}\cdots a_1} - K'^{\nu}_{a_{r+1}\cdots a_1} \doteq \epsilon\Big[D_{a_{r+1}}L^{\nu}_{a_r\cdots a_1}$$

$$+ (\nabla_{a_{r+1}}V^{\rho})\nabla_{\rho}K^{\nu}_{a_r\cdots a_1} - \sum_{1}^{r}{}_{s}Q^{\sigma}_{a_s a_{r+1}}K^{\nu}_{a_r\cdots a_{s+1}\sigma a_{s-1}\cdots a_1} \tag{6,6}$$

$$+ V^{\omega}(\partial_{a_{r+1}}\partial_{\omega} - \partial_{\omega}\partial_{a_{r+1}})K^{\nu}_{a_r\cdots a_1}\Big] + \cdots .$$

Since

$$\begin{aligned} (\partial_{a_{r+1}}\partial_{\omega} - \partial_{\omega}\partial_{a_{r+1}})K^{\nu}_{a_r\cdots a_1} \\ &= [T^{\mu}_{a_{r+1}}\partial_{\mu}\partial_{\omega} - (\partial_{\omega}T^{\mu}_{a_{r+1}})\partial_{\mu} - T^{\mu}_{a_{r+1}}\partial_{\omega}\partial_{\mu}]K^{\nu}_{a_r\cdots a_1} \\ &\doteq -(\nabla_{\omega}T^{\mu}_{a_{r+1}})\nabla_{\mu}K^{\nu}_{a_r\cdots a_1}, \end{aligned}$$

the equation (6,6) reduces to

$$
\begin{aligned}
{}^{*}K^{\nu}_{a_{r+1}\cdots a_1} \quad K'^{\nu}_{a_{r+1}\cdots a_1} \doteq \epsilon\Big[& D_{a_{r+1}}L^{\nu}_{a_r\cdots a} \\
& + L^{\rho}_{a_{r+1}}\nabla_{\rho}K^{\nu}_{a_r\cdots a_1} - \sum_{1}^{r}{}_{s}\mathcal{Q}^{c}_{a_s a_{r+1}}K^{\nu}_{a_r\cdots a_{s+1}ca_{s-1}\cdots a_1}\Big] + \cdots .
\end{aligned} \tag{6,7}
$$

In this tensor equation we may obviously suppress the dot in $\doteq$ and consequently if we put

$$
\begin{aligned}
L^{\nu}_{a_{r+1}\cdots a_1} \equiv\ & D_{a_{r+1}}L^{\nu}_{a_r\cdots a_1} + L^{\rho}_{a_{r+1}}\nabla_{\rho}K^{\nu}_{a_r\cdots a_1} \\
& - \sum_{1}^{r}{}_{s}\mathcal{Q}^{c}_{a_s a_{r+1}}K^{\nu}_{a_r\cdots a_{s+1}ca_{s-1}\cdots a_1},
\end{aligned}
$$

we have from (6,7)

$$
\Delta K^{\nu}_{a_{r+1}\cdots a_r} = L^{\nu}_{a_{r+1}\cdots a_1}.
$$

Hence if (6,2) holds for some r then it holds also for $r+1$. But (6,2) holds for $r=2$ and consequently it holds for $r=2, 3, \cdots$. The remaining part of the theorem (equation (6,1)b) may be proved by means of an equation similar to (5,10)

7. **The differential equation $L^{\nu}_{a}=0$.**

THEOREM (7,1). *The system of differential equations*

$$
L^{\nu}_{a} = 0 \quad \text{or} \quad \nabla_{a}V^{\nu} = V^{\lambda}\nabla_{\lambda}T^{\nu}_{a} \tag{7,1}
$$

is completely integrable and its general solution involves n arbitrary constants.

PROOF. We have first

$$
\begin{aligned}
\nabla_{[b}\nabla_{a]}V^{\nu} &= \nabla_{[b}T^{\mu}_{a]}\nabla_{\mu}V^{\nu} = (\nabla_{[b}T^{\mu}_{a]})\nabla_{\mu}V^{\nu} + T^{\omega}_{[b}T^{\mu}_{a]}\nabla_{\omega}\nabla_{\mu}V^{\nu} \\
&= -\frac{1}{2}R^{\nu}_{\omega\mu\lambda}V^{\lambda}T^{\omega\mu}_{ba}
\end{aligned} \tag{7,2}
$$

and by virtue of (7,1)

$$
\begin{aligned}
\nabla_{[b}V^{\lambda}\nabla_{|\lambda|}T^{\nu}_{a]} &= (\nabla_{[b}V^{\lambda})\nabla_{|\lambda|}T^{\nu}_{a]} + V^{\mu}T^{\omega}_{[b}\nabla_{|\omega}\nabla_{\mu|}T^{\nu}_{a]} \\
&= V^{\omega}[(\nabla_{\omega}T^{\mu}_{[b})(\nabla_{|\mu|}T^{\nu}_{a]}) + T^{\mu}_{[b}\nabla_{|\mu}\nabla_{|\omega}T^{\nu}_{a]}].
\end{aligned} \tag{7,3}
$$

On the other hand

$$
0 = V^{\omega}\nabla_{\omega}T^{\mu}_{[b}\nabla_{|\mu|}T^{\nu}_{a]} = V^{\omega}(\nabla_{\omega}T^{\mu}_{[b})\nabla_{|\mu|}T^{\nu}_{a]} + V^{\omega}T^{\mu}_{[b}\nabla_{\omega|}\nabla_{\mu|}T^{\nu}_{a]}
$$

and consequently (7,3) reduces to

$$(7,4)\qquad \nabla_{[b}V^{\lambda}\nabla_{|\lambda|}T^{\nu}_{a]} = V^{\omega}T^{\mu}_{[b}(\nabla_{|\mu}\nabla_{\omega|} - \nabla_{|\omega}\nabla_{\mu|})T^{\nu}_{a]} = V^{\omega}T^{\mu\lambda}_{[ba]}R^{\nu}_{\omega\mu\lambda}.$$

The integrability conditions of (7,1) are obtained by substituting in

$$\nabla_{[b}\nabla_{a]}V^{\nu} = \nabla_{[b}V^{\lambda}\nabla_{|\lambda|}T^{\nu}_{a]}$$

from (7,2) and (7,4). In doing so we obtain

$$\frac{1}{2}V^{\lambda}T^{\omega\mu}_{ba}(-R^{\nu}_{\omega\mu\lambda} - R^{\nu}_{\lambda\omega\mu} + R^{\nu}_{\lambda\mu\omega}) = -\frac{1}{2}V^{\lambda}T^{\omega\mu}_{ba}(R^{\nu}_{\omega\mu\lambda} + R^{\nu}_{\lambda\omega\mu} + R^{\nu}_{\mu\lambda\omega}) \equiv 0$$

since

$$R^{\nu}_{\omega\mu\lambda} + R^{\nu}_{\mu\lambda\omega} + R^{\nu}_{\lambda\omega\mu} \equiv 0.$$

Hence the integrability conditions of (7,1) are identically satisfied and consequently (7,1) is completely integrable and its general solution involves n arbitrary constants.

THEOREM (7,2). *Let $V^{\nu}_{1}, \cdots, V^{\nu}_{n}$ be n linear independent solutions of* (7,1). *Then*[10]

$$(7,5)\qquad W^{\nu} = V^{\mu}_{[\mathfrak{a}}\partial_{\mu}V^{\nu}_{\mathfrak{e}]} \quad (\mathfrak{a}, \mathfrak{e} = 1, \cdots, n;\ \mathfrak{a} \neq \mathfrak{e})$$

is a vector solution of (7,1).

PROOF. W^{ν} is obviously a vector. Hence to prove our theorem we have to show that it satisfies the equation

$$(7,6)\qquad \partial_{a}W^{\nu} = W^{\lambda}\partial_{\lambda}T^{\nu}_{a}.$$

From (7,1) we have for

$$(7,7)\qquad \partial_{a}W^{\nu} = \partial_{a}V^{\mu}_{[\mathfrak{a}}\partial_{\mu}V^{\nu}_{\mathfrak{e}]} = V^{\lambda}_{[\mathfrak{a}}(\partial_{\lambda}T^{\mu}_{a})\partial_{\mu}V^{\nu}_{\mathfrak{e}]} + V^{\mu}_{[\mathfrak{a}}\partial_{a}\partial_{\mu}V^{\nu}_{\mathfrak{e}]}.$$

On the other hand $(\partial_{a}\partial_{\mu} - \partial_{\mu}\partial_{a})V^{\nu}_{\mathfrak{e}} = -(\partial_{\mu}T^{\lambda}_{a})\partial_{\lambda}V^{\nu}_{\mathfrak{e}}$ and consequently (7,7) reduces to

$$(7,8)\qquad \begin{aligned}\partial_{a}W^{\nu} &= (\partial_{\lambda}T^{\mu}_{a})(V^{\lambda}_{[\mathfrak{a}}\partial_{\mu}V^{\nu}_{\mathfrak{e}]} - V^{\lambda}_{[\mathfrak{a}}\partial_{\mu}V^{\nu}_{\mathfrak{e}]}) + V^{\mu}_{[\mathfrak{a}}\partial_{\mu}\partial_{a}V^{\nu}_{\mathfrak{e}]}\\ &= V^{\mu}_{[\mathfrak{a}}\partial_{\mu}V^{\lambda}_{\mathfrak{e}]}\partial_{\lambda}T^{\nu}_{a} = V^{\mu}_{[\mathfrak{a}}(\partial_{\mu}V^{\lambda}_{\mathfrak{e}]})\partial_{\lambda}T^{\nu}_{a} = W^{\lambda}\partial_{\lambda}T^{\nu}_{a}.\end{aligned}$$

[10] It is assumed that in the equations (7,5)–(7,8) the brackets [ ] do not affect the Greek and Latin indices.

REMARK. The theorem (7,2) shows that the set of all generators $V^\lambda\partial_\lambda$ of infinitesimal transformations (2,1) is closed under the operations of taking linear combination and forming commutators. Hence these generators form a Lie algebra.

8. Special cases.

DEFINITION (8,1). *We say that a transformation* (2,1) *reproduces the family* (V_m) *if*

$$\Delta K^r_{a_r\cdots a_1} = 0 \qquad (r = 1, 2, \cdots; K^r_a \equiv T^r_a). \tag{8,1}$$

THEOREM (8,1). *A necessary and sufficient condition that* (V_m) *be reproduced under* (2, 1) *is:* V^r *is a solution of*

$$L^r_a = 0. \tag{8,2}$$

PROOF. Let (V_m) be reproduced by (2,1). Then from the first of (8,1) we have (8,2) by virtue of (3,1)c. Conversely, if (8,2) is satisfied, the following equations hold (cf. (3,4)c and (4,8))

$$\text{(a)} \quad {}^*g_{ab} = g'_{ab} + \cdots, \qquad \text{(b)} \quad {}'\Gamma^e_{ab} = \Gamma^{e\prime}_{ab} + \cdots. \tag{8,3}$$

Consequently we have from (4,1)c and (8,3)b

$$\text{(c)} \quad {}^*\Gamma^e_{ab} = {}'\Gamma^e_{ab} + \epsilon Q^e_{ab} + \cdots. \tag{8,3}$$

On the other hand we get by virtue of (8,3)a and (8,3)b

$$\text{(d)} \quad {}^*\Gamma^e_{ab} = {}'\Gamma^e_{ab} + \cdots. \tag{8,3}$$

Comparing (8,3)(c) and (d) we have

$$Q^e_{ab} = 0. \tag{8,4}$$

The equations (8,2), (8,4) together with (6,2) and (6,1)a lead to (8,1) for $r = 2, 3, \cdots$.

In the next theorem we shall consider a family (V_m) of totally geodesic subspaces, that is, a family for which

$$K^r_{ab} = 0 \tag{8,5}$$[11]

and we prove first the following lemma.

LEMMA (8,1). *If* (8,5) *holds, then* W_{abd} (*cf. the equation* (4,2)) *satisfies*

[11] This equation imposes some conditions on the structure of the large space V_n (cf. the equation (8,12)).

the following equation

$$W_{bad} \equiv W_{abd} = P^{c}_{bda} W_c + D_a D_b W_d \tag{8,6}$$

where P^{c}_{bda} *is the curvature tensor of* V_m *and* $W_c \equiv T^{\lambda}_{c} V_\lambda$.

PROOF. Let $W^\nu (N^\nu)$ be the tangential (normal) component of V^ν

$$V^\nu = W^\nu + N^\nu, \qquad W^\nu = V^\lambda T^{c}_{\lambda} T^{\nu}_{c}, \qquad N_\nu T^{\nu}_{c} = 0. \tag{8,7}$$

From (3,5) we have

$$\begin{aligned} W_{ab} = T^{\lambda}_{(a} T^{\mu}_{b)} \nabla_\lambda V_\mu &= T^{\lambda\mu}_{(ab)} \nabla_\lambda W_\mu + T^{\lambda\mu}_{(ab)} \nabla_\lambda N_\mu \\ &= D_{(a} W_{b)} - N_\mu \nabla_{(a} T^{\mu}_{b)}. \end{aligned} \tag{8,8a}$$

If (8,5) holds then

$$0 = N_\mu D_a T^{\mu}_{b} = N_\mu (\nabla_a T^{\mu}_{b} - \Gamma^{c}_{ba} T^{\mu}_{c}) = N_\mu \nabla_a T^{\mu}_{b}$$

and (8,8)a reduces to

$$W_{ab} = D_{(a} W_{b)}. \tag{8,8b}$$

Consequently, we get from (4,2)a

$$W_{abd} = D_{[a} D_{d]} W_b + D_{[b} D_{d]} W_a + D_{[b} D_{a]} W_d + D_a D_b W_d. \tag{8,9a}$$

Because for any vector P_b whatsoever

$$D_{[a} D_{d]} P_b + D_{[d} D_{b]} P_a + D_{[b} D_{a]} P_d = 0$$

$$D_{[b} D_{d]} P_a = 1/2 P^{c}_{bda} P_c$$

where P^{c}_{dba} is the curvature tensor of V_m, (8,9)a reduces to (8,6).

DEFINITION (8,2). *If* $A = 0$ *is an analytic expression for a property of* (V_m), *then we say that* (2,1) *preserves this property if* $\Delta A = 0$.

THEOREM (8,2). *Let* (V_m) *be a family of totally geodesic subspaces. A necessary and sufficient condition that this property be preserved by* (2,1) *is: The vector* V^ν *is a solution of any one of the following equations*

$$\begin{aligned} &\text{(a)} \quad L^{\nu}_{ab} = 0, \\ &\text{(b)} \quad D_b D_a V^\nu - V^\omega T^{\mu\lambda}_{ba} R^{\nu}_{\omega\mu\lambda} - W^{c}_{ba} T^{\nu}_{c} = 0, \\ &\text{(c)} \quad D_b D_a N^\nu - N^\omega T^{\mu\lambda}_{ba} R^{\nu}_{\omega\mu\lambda} = 0, \end{aligned} \tag{8,10}$$

where in the last equation

$$N^\nu = V^\lambda (\delta^{\nu}_{\lambda} - T^{\nu}_{c} T^{c}_{\lambda}).$$

A sufficient but not necessary condition is:

The vector V^r is a solution of (8,2). *If* (8,10) *but not* (8,2) *is satisfied by V^r, then (V_m) is not reproduced. If* (8,2) *holds, then* (8,10) *is satisfied and (V_m) is reproduced.*

PROOF. According to our assumption the transformation (2,1) carries a family (V_m) in a family (V_m^*). Hence in order to prove our theorem we have only to look for conditions that the equation $K_{ab}^r = 0$ be satisfied. The necessity and sufficiency of (8,10)a follows at once from (5,8). If (8,5) is satisfied, then we have

$$K_{ab}^{\prime r} = K_{ab}^r + \epsilon V^\lambda \partial_\lambda K_{ab}^r + \cdots = 0 + \cdots \tag{8,11)a}$$

and consequently

$$\Delta K_{ab}^r = (\tau - \omega) K_{ab}^r = \tau K_{ab}^r. \tag{8,11)b}$$

Hence, we have from (8,11)b, (5,8) and (5,2) that in our case a necessary and sufficient condition for $\Delta K_{ab}^r = 0$ is that V^r be a solution of (8,10)b and this equation is (in our case) equivalent to (8,10)a. On the other hand, the integrability conditions of (8,5) are

$$P_{cba}^d T_d^r = T_{cba}^{\omega\mu\lambda} R_{\omega\mu\lambda}^r \tag{8,12}$$

and consequently $T_{cba}^{\omega\mu\lambda} R_{\omega\mu\lambda}^r$ is in (V_m). If we decompose V^r according to (8,7), then we have from (8,12)

$$\begin{aligned} -V^\omega T_{ba}^{\mu\lambda} R_{\omega\mu\lambda}^r &= -W^c T_{cba}^{\omega\mu\lambda} R_{\omega\mu\lambda}^r - N^\omega T_{ba}^{\mu\lambda} R_{\omega\mu\lambda}^r \\ &= -W^c P_{cba}^d T_d^r - N^\omega T_{ba}^{\mu\lambda} R_{\omega\mu\lambda}^r. \end{aligned} \tag{8,13)a}$$

On the other hand, if (8,5) holds then

$$D_b D_a V^r = D_b D_a W^c T_c^r + D_b D_a N^r = (D_b D_a W^c) T_c^r + D_b D_a N^r. \tag{8,13)b}$$

Furthermore, if (8,5) holds we have according to (8,6)

$$\begin{aligned} -W_{ba}^c T_c^r &= -P_{bda} W_e g^{dc} T_c^r - (D_a D_b W^c) T_c^r \\ &= (P_{eab}^c W^e - D_a D_b W^c) T_c^r. \end{aligned} \tag{8,13)c}$$

[12]

Comparing (8,10)b with (8,13) we have

$$\begin{aligned} &D_b D_a V^r - V^\omega T_{ba}^{\mu\lambda} R_{\omega\mu\lambda}^r - W_{ba}^c T_c^r \\ &\quad = T_d^r [D_b D_a W^d - W^c P_{cba}^d + P_{cab}^d W^c - D_a D_b W^d] \\ &\quad\quad + D_b D_a N^r - N^\omega T_{ba}^{\mu\lambda} R_{\omega\mu\lambda}^r = 0. \end{aligned} \tag{8,14)a}$$

[12] We use here the well known identities $P_{bcae} = P_{aebc} = -P_{eabc}$.

Since

$$(D_bD_a - D_aD_b)W^d = -P^d_{bac}W^c$$
$$= P^d_{abc}W^c = -(P^d_{bca} + P^d_{cab})W^c$$

we obtain from (8,14)a

(8,14)b $$D_bD_aV^\nu - V^\omega T^{\mu\lambda}_{ba}R^\nu_{\omega\mu\lambda} - W^c_{ba}T^\nu_c = D_bD_aN^\nu - N^\omega T^{\mu\lambda}_{ba}R^\nu_{\omega\mu\lambda}$$

and this equation proves the statement about (8,10)c. The last part of the theorem follows easily from Theorem (8,1).

REMARK I. The statement about the equation (8,10)b is a generalization of Levi-Civita's result for $m=1$.[13]

REMARK II. Let $m=1$, $n=2$. Then

(8,15)
$$R^\nu_{\omega\mu\lambda} = 2K g_{\lambda[\omega}\delta^\nu_{\mu]},$$
$$N^\omega T^\mu_1 R^\lambda_1 R^\nu_{\omega\mu\lambda} = -KN^\nu T \qquad (T = g_{\lambda\mu}T^\lambda_1 T^\mu_1).$$

If $\eta^1 \equiv s$, s being the arc of V_1, then $D_1 = T^\lambda_1\nabla_\lambda$ and $T=1$. Moreover, if j^ν is the unit vector in the direction of $D_1T^\nu_1$, then

(8,16)
$$D_1T^\nu_1 = j^\nu k, \qquad D_1j^\nu = -kT^\nu_1,$$

where k is the curvature of V_1. Because we may put $N^\nu = yj^\nu$, y being a factor of proportionality, we have on account of (8,16)

(8,17)
$$D_1N^\nu = y'j^\nu - ykT^\nu_1.$$

Consequently

$$D_1^2N^\nu = y''j^\nu - 2y'kT^\nu_1 - yk'T^\nu_1 - yk^2j^\nu.$$

If (V_1) consists of geodesic lines, we have $k=0$ and

(8,18)a
$$j_\nu D_1^2N^\nu = y''.$$

On the other hand, we obtain from (8,15) for $\eta^1 = s$

(8,18)b
$$j_\nu N^\omega T^\mu_1 T^\lambda_1 R^\nu_{\omega\mu\lambda} = -Ky.$$

Comparing (8,10)c and (8,18) we have

$$y'' + Ky = 0,$$

[13] Levi-Civita, T., *Sullo scostamento geodetico* (Bollettino della Unione Matematica Italiana (1926) pp. 1–4) and *Sur l'écart géodésique* (Math. Ann. vol. 97 (1926) pp. 292–320). For more general metric spaces cf. also Hlavatý, V., *Les courbes de la variété générale à n dimensions* (Mémorial des Sciences Mathématiques, no. 42).

which is the well known equation found by Jacobi.

Bibliography

1. J. A. Schouten, *On infinitesimal deformations of V_m in V_n*, Proc. K. Akademie van Wetenschappen vol. 31 (1928) pp. 208–218.

2. E. Bortolotti, *Scostamento geodetico e sue generalizzazioni*, Giornale di matematiche vol. 66 (1928) pp. 153–191.

3. A. G. Walker, *On small deformation of sub-spaces of a flat space*, Proceedings of the Edinburgh Mathematical Society vol. 3 (1932) pp. 77–86.

4. J. A. Schouten and E. R. van Kampen, *Beiträge zur theorie der deformation*, Prace Matematyczno-Fizyczne Warszawa vol. 41 (1933) pp. 1–19.

5. H. A. Hayden, *Infinitesimal deformations of sub-spaces in a general metrical space*, Proc. London Math. Soc. vol. 37 (1934) pp. 416–440.

6. ———, *Infinitesimal deformation of an L_m in an L_n*, Proc. London Math. Soc. vol. 41 (1936) pp. 332–336.

7. J. A. Schouten and D. J. Struik, *Einführung in die neueren Methoden der Differentialgeometrie*, vol. 1, Groningen, 1935, p. 151.

8. E. T. Davies, *On the deformation of a subspace*, J. London Math. Soc. vol. 11 (1936) pp. 295–301.

9. ———, *On the second and third fundamental forms of a subspace*, J. London Math. Soc. vol. 12 (1937) pp. 290–295.

10. P. Dienes and E. T. Davies, *On the infinitesimal deformations of tensor submanifolds*, Journal de Mathématiques vol. 16 (1937) pp. 111–150.

11. K. Yano, *Sur les déformations infinitésimales des sous-espaces dans un espace affine*, Proc. Imp. Acad. Tokyo vol. 21 (1945) pp. 248–260.

12. ———, *Sur la déformation infinitésimale tangentielle d'un sous-espace*, Proc. Imp. Acad. Tokyo vol. 21 (1945) pp. 261–268.

Indiana University

REMARKS ON CERTAIN PATHOLOGICAL OPEN SUBSETS OF 3-SPACE AND THEIR FUNDAMENTAL GROUPS

W. A. BLANKINSHIP AND R. H. FOX

We shall consider several well known subsets of spherical 3-space S: the compact zero-dimensional set P described by Antoine [1][1] and the topological 3-cell C consisting of the "horned sphere" Σ of Alexander [2] together with its interior. That *the fundamental groups, $\pi_1(S-P)$ and $\pi_1(S-C)$, of their complementary domains are not finitely generated* (and hence in particular not trivial) was noted by Alexander [2, pp. 9 and 11]. It is the main purpose of this note to give a

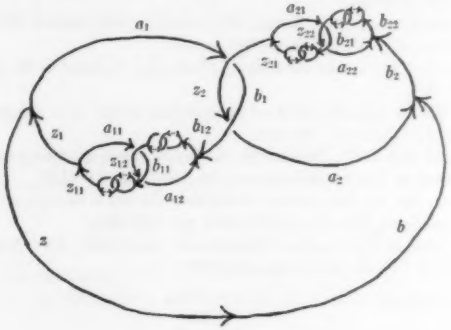


FIG. 1

complete proof of this remark, using explicit presentations of the groups.[2] We feel that this is justified by the historical significance of these examples, even though on the one hand Antoine's set occurs as a special case of a recent generalization [3] to n-dimensional space ($n \geqq 3$) and on the other Alexander's example has been supplanted by a much simpler one [4]. An unexpected consequence of the explicit presentation of the groups is the discovery, of which we make use, that $\pi_1(S-C)$ is a homomorph of $\pi_1(S-P)$; the geometrical meaning of this relationship between Antoine's and Alexander's examples is not evident.

Of great interest, although not as well known, is the simply con-

Received by the editors July 13, 1949.

[1] Numbers in brackets refer to the references cited at the end of the paper.

[2] A proof that $\pi_1(S-P)$ is non-trivial has been given by R. P. Coelho [6]. By a presentation $\{x/r=1\}$ of a group we mean a system of generators $\{x_i\}$ and defining relations $\{r_i=1\}$.

nected open subset M of S, which is not a topological open 3-cell, constructed by Whitehead and Newman [5]. In §3 we give a simplified proof that M is not homeomorphic to an open 3-cell. The simplifying ideas in this proof were used [4, example 1.3] in demonstrating the properties of a certain set Z which combines the essential features of the example of Alexander and that of Whitehead-Newman.

1. Alexander's horned cell. The complement, $S-C$, of Alexander's horned cell is homeomorphic to the complement of an infinite graph Γ, whose projection is shown in Fig. 1.

From this diagram a presentation of the group $\pi_1(S-C)=\pi_1(S-\Gamma)$ may be read, using a slight extension of a standard method.[3] The generators are z, z_α, a_α, b, b_α, where α ranges over all finite sequences of the form

$$\alpha = \alpha_1\alpha_2 \cdots \alpha_l, \qquad \alpha_i = 1, 2, \qquad l(\alpha) = l > 0.$$

The defining relations are of three types:

(1) At the crossings (cf. Fig. 2)

(1_α) $$b_{\alpha 1} = z_{\alpha 2} a_{\alpha 1} z_{\alpha 2}^{-1}, \qquad z_{\alpha 2} = b_{\alpha 1}^{-1} a_{\alpha 2} b_{\alpha 1}.$$

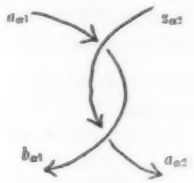


FIG. 2

(2) At the points of order three (cf. Fig. 3)

(2_α) $$z_\alpha^{-1} z_{\alpha 1} a_{\alpha 1}^{-1} = 1, \qquad b_\alpha b_{\alpha 2}^{-1} a_{\alpha 2} = 1.$$

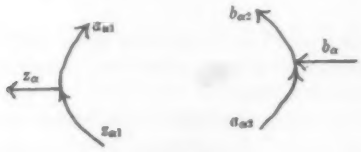


FIG. 3

(3) Around the singular points (cf. Fig. 4)

(3_α) $$b_\alpha = z_\alpha,$$

[3] Cf. [4, p. 981].

where $\alpha 1$, $\alpha 2$ denote the sequences of length $l+1$ formed by adjoining 1 or 2, resp., to α.

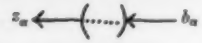


FIG. 4

Using (2_α) and (3_α) as definitions of the generators a and b we obtain from (1_α) the presentation:[4]

$$G = \pi_1(S - C) = \{z_\alpha, l(\alpha) \geqq 0 / z_\alpha = [z_{\alpha 1}, z_{\alpha 2}^{-1}]\}.$$

For each non-negative integer n we consider the free group G_n on generators z_α, $l(\alpha) = n$, and define a homomorphism ϕ_n of G_n into G_{n+1} by $\phi_n(z_\alpha) = [z_{\alpha 1}, z_{\alpha 2}^{-1}]$. Clearly G is the direct limit of the homomorphism sequence $G_0 \to G_1 \to G_2 \to \cdots$. Moreover, for each n, ϕ_n is an isomorphism, for if $w(z_\alpha)$ is a reduced word in G_n, then $\phi_n(w(z_\alpha)) = w([z_{\alpha 1}, z_{\alpha 2}^{-1}])$ is necessarily a reduced word in G_{n+1}. Hence the groups G_n may be identified with certain subgroups of G. Thus $G = \bigcup_{n=0}^{\infty} G_n$ is the union of an increasing sequence of finitely generated free groups. Since the rank of G_n is 2^n we obtain the result,[5]

$\pi_1(S-C)$ is a locally free group and does not have finite rank.

2. **Antoine's set.** Consider a solid torus P_0 in S and k solid tori $T_1, T_2, \cdots, T_k$ in P_0 arranged in cyclic order around P_0 in such a way that each torus is simply linked with each of its neighbors. Denote the union of $T_1, T_2, \cdots, T_k$ by P_1 and for each $i = 1, 2, \cdots, k$ select a homeomorphism f_i of P_0 onto T_i. For any finite sequence $\alpha = \alpha_1 \alpha_2 \cdots \alpha_l$, $1 \leqq \alpha_i \leqq k$, define $f_\alpha = f_{\alpha 1} f_{\alpha 2} \cdots f_{\alpha l}$, $T_\alpha = f_\alpha(P_0)$, and $P_l = \bigcup_{l(\alpha)=l} T_\alpha$. Assuming k sufficiently large and the tori $T_1, \cdots, T_k$ properly placed so that diam $T_\alpha \to 0$ as $l(\alpha) \to \infty$, the set $P = \bigcap_{l=0}^{\infty} P_l$ is recognizable as Antoine's set. We denote the closure of the set $P_l - P_n$ by P_n^l.

It is easily seen that P_1^0 is of the same homotopy type as the complement of the linkage whose projection is represented in Fig. 5, where we have taken k even.

Labelling and orienting the edges as indicated we find by a standard procedure[3] that $\pi_1(P_1^0)$ is generated by elements $x, u, v, w, a_1, b_1, \cdots, a_k, b_k$ subject to the defining relations[4]

---

[4] We use the notations $[u, v] = uvu^{-1}v^{-1}$, $u^v = vuv^{-1}$, $u^{-v} = vuv^{-1}v^{-1}$.

[5] To show just that $\pi_1(S-C)$ is non-trivial, a simpler procedure is to exhibit a representation of G into the symmetric group of order 6. Cf. [3]. For definition of locally free group see [8].

$$
\begin{aligned}
x^{a_1} &= w = x^{b_1}, \\
b_1^{x} &= v = u^{b_k^{-1}}, \\
u^{x^{-1}} &= a_1 = b_1^{b_2}, \\
a_2^{a_1} &= b_2 = a_2^{a_3}, \\
b_3^{b_2} &= a_3 = b_3^{b_4}, \\
a_4^{a_3} &= b_4 = a_4^{a_5}, \\
&\cdots\cdots\cdots, \\
b_{k-1}^{b_{k-2}} &= a_{k-1} = b_{k-1}^{b_k}, \\
a_k^{a_{k-1}} &= b_k = a_k^{u}.
\end{aligned}
$$

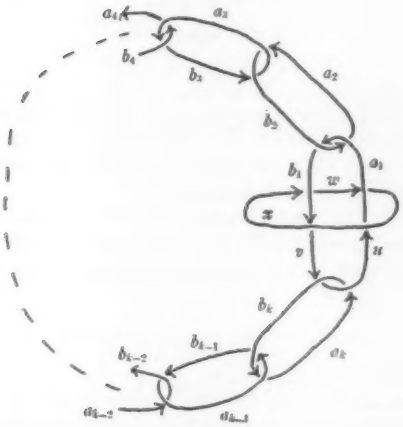


FIG. 5

From these relations we can deduce that

$$a_1^{-1}b_1 = a_2^{-1}b_2 = \cdots = a_k^{-1}b_k = u^{-1}v.$$

Denoting this element by y, we are now enabled to eliminate $u, v, w, b_1, \cdots, b_k$. As a result, we find that $\pi_1(P_1^0)$ is generated by $x, y, a_1, \cdots, a_k$, subject to the relations

$$
\text{I} \qquad
\begin{aligned}
&[x, y] = 1, \\
&y = [a_j^{-1}, a_{j-1}] = [a_j^{-1}, a_{j+1}], \qquad j = 2, 4, 6, \cdots, k-2, \\
&y = [a_k^{-1}, a_{k-1}] = [a_k^{-1}, a_1^{x}].
\end{aligned}
$$

Now since the closure of the set $T_i - U_j T_{ij}$ is just $f_i(P_1^0)$, we see that its fundamental group is given by generators x_i', y_i', a_{i1}', a_{i2}', $\cdots$, a_{ik}'

and defining relations I_i', obtained by replacing x, y, $a_1, \cdots, a_k$ in I by x_i', y_i', $a_{i1}', \cdots, a_{ik}'$, respectively. The intersection of P_1^0 and $f_i(P_1^0)$ is the torus $\dot{T}_i$ which is the boundary of T_i, and whose fundamental group is free abelian on the two generators x_i' and y_i'. Since x and y may be represented by loops bounding in the closure of $S-P_0$ and in P_0, respectively (see Fig. 5), it follows that $x_i' = f_i(x)$ and $y_i' = f_i(y)$ may be represented respectively by equatorial and meridional loops on T_i. This observation enables us to compute from Fig. 5 the relations defining the injection $\pi_1(\dot{T}_i) \to \pi_1(P_1^0)$. They are

$$\mathrm{II}_i \qquad \begin{aligned} y_i &= a_i, \\ x_i &= a_{i+1}^{-1} a_{i-1}, && i = 2, 4, \cdots, k-2, \\ &= a_{i+1} a_{i-1}^{-1}, && i = 3, 5, \cdots, k-1, \\ &= a_2 x^{-1} a_k^{-1} x, && i = 1, \\ &= x a_1^{-1} x^{-1} a_{k-1}, && i = k, \end{aligned}$$

where x_i, y_i (and later $a_{i1}, \cdots, a_{ik}$) denote the elements represented by loops obtained by conjugating a representative of x_i', y_i' ($a_{i1}', \cdots, a_{ik}'$) by a (properly chosen) fixed path joining the base point of $\pi_1(P_1^0)$ to that of $\pi_1(f_i(P_1^0))$.

Since the k sets $f_i(P_1^0)$ are mutually disjoint, we may obtain a presentation of the group $\pi_1(P_2^0)$ by k successive applications of the algorithm for computing the fundamental group of a union.[6] We thus obtain

$$\pi_1(P_2^0) = \{x_\alpha, y_\alpha, a_{\alpha i}/\mathrm{I}, \mathrm{I}_i, \mathrm{II}_i\}$$

where $0 \leqq l(\alpha) \leqq 1$, $i = 1, 2, \cdots, k$.

Proceeding similarly with the adjunction of the sets P_3^2, P_4^3, $\cdots$, we construct inductively presentations of the groups $\pi_1(P_n^0)$:

$$\pi_1(P_n^0) = \{x_\alpha, y_\alpha, a_{\alpha i}/\mathrm{I}, \mathrm{I}_{\alpha i}, \mathrm{II}_{\alpha i}\}$$

where $0 \leqq l(\alpha) \leqq n-1$, $i = 1, 2, \cdots, k$ and $\mathrm{I}_{\alpha i}$, $\mathrm{II}_{\alpha i}$ denote the relations obtained from I_i and II_i by replacing i by αi (see below).

Since $\pi_1(P_0-P)$ is the direct limit of the homomorphism sequence

$$\pi_1(P_1^0) \to \pi_1(P_2^0) \cdots,$$

in which each homomorphism sends the generators of one group into the same-named generators of the other, we obtain $\pi_1(P_0-P)$ simply by removing the restrictions on $l(\alpha)$ in the presentations. A presentation of the group $\pi_1(S-P)$ is obtained by adjoining the relation $x=1$.

[6] Cf. [7, §52].

Thus $\pi_1(S-P)$ is given by generators x_α, y_α, $a_{\alpha i}$, $l(\alpha) \geqq 0$, with the defining relations:

$$\mathrm{I}_\alpha \quad \begin{aligned} & x = 1, \\ & [x_\alpha, y_\alpha] = 1, \\ & y_\alpha = [a_{\alpha j}^{-1}, a_{\alpha j-1}] = [a_{\alpha j}^{-1}, a_{\alpha j+1}], && j = 2, 4, \cdots, k-2, \\ & y_\alpha = [a_{\alpha k}^{-1}, a_{\alpha k-1}] = [a_{\alpha k}^{-1}, a_{\alpha 1}^{x_\alpha}], \end{aligned}$$

$$\mathrm{II}_{\alpha i} \quad \begin{aligned} & y_{\alpha i} = a_{\alpha i}, \\ & x_{\alpha i} = a_{\alpha i+1}^{-1} a_{\alpha i-1}, && i = 2, 4, \cdots, k-2, \\ & \quad = a_{\alpha i+1}^{-1} a_{\alpha i-1}, && i = 3, 5, \cdots, k-1, \\ & \quad = a_{\alpha 2}^{-1} x_\alpha^{-1} a_{\alpha k} x_\alpha, && i = 1, \\ & \quad = x_\alpha a_{\alpha 1}^{-1} x_\alpha^{-1} a_{\alpha k-1}, && i = k. \end{aligned}$$

That $\pi_1(S-P)$ is not finitely generated (and therefore nontrivial) is demonstrated by the following homomorphism into $\pi_1(S-C)$:

$$\begin{aligned} a_\alpha = y_\alpha &\to z_{\sigma(\alpha)}, && \text{if } l(\alpha) \text{ is odd,} \\ &\to z_{\sigma(\alpha)}^{-1}, && \text{if } l(\alpha) \text{ is even,} \end{aligned}$$

where

$$\sigma(\alpha) = \sigma_1 \sigma_2 \cdots \sigma_l, \qquad \sigma_i = 1, 2,$$

for $i = 1, 2, \cdots, l$ and

$$\begin{aligned} \sigma_i &\equiv \alpha_i \pmod 2 && \text{if } i \text{ is odd,} \\ &\equiv 1 + \alpha_i \pmod 2 && \text{if } i \text{ is even.} \end{aligned}$$

3. **The Whitehead-Newman set.** To obtain the Whitehead-Newman set T_∞ one proceeds as in the construction of Antoine's set, using $k=1$ instead of $k \equiv 0 \pmod 2$. The solid torus corresponding to $S-P_0$ of the preceding section is denoted in [5] by $R=S-T_0$. The fundamental group G_∞ of $U_{0,\infty}=T_0-T_\infty=S-(R \cup T_\infty)$ is the limit of the direct homomorphism sequence

$$G_1 \to G_2 \to G_3 \to \cdots$$

where G_n denotes the fundamental group of $U_{0,n}=T_0-T_n=S-(R \cup T_n)$. From the presentation (3.1) of [5] it follows that G_∞ has the presentation

$$G_\infty = \{a_\lambda, b_\lambda, 0 \leqq \lambda < \infty / [a_\lambda, b_\lambda] = 1, b_\lambda = [a_\lambda^{-1}, b_{\lambda+1}][a_\lambda^{-1}, b_{\lambda+1}^{-1}], \\ a_{\lambda+1} = [b_{\lambda+1}^{-1}, a_\lambda][b_{\lambda+1}^{-1}, a_\lambda^{-1}]\}.$$

The fundamental group of $S-T_\infty$, obtained by adjoining to G_∞ the relation $a_0=1$, is obviously trivial. Thus $S-T_\infty$ is a simply connected open subset of S.

On the other hand the group G_∞ has the representation

$$\begin{aligned} a_\lambda &\to (1\ 2\ 3\ 4\ 5) && \text{(for } \lambda \text{ even)} \\ &\to (1\ 5\ 3\ 2\ 4) && \text{(for } \lambda \text{ odd)}, \\ b_\lambda &\to (5\ 4\ 3\ 2\ 1) && \text{(for } \lambda \text{ even)} \\ &\to (4\ 2\ 3\ 5\ 1) && \text{(for } \lambda \text{ odd)}, \end{aligned}$$

in the symmetric group of order 5.

We assert that the complement $S-(T_\infty \cup J)$ of any compact subset J of $S-T_\infty$ containing R cannot be simply connected. For $S-(T_\infty \cup J) \subset S-(R \cup T_\infty) \subset U_{0,\infty}$, and we can always find λ so large that a_λ is represented by a loop in $U_{0,\infty}-J$, and this cannot be contracted to a point in $S-(T_\infty \cup J)$ because $a_\lambda \neq 1$ in $\pi_1(U_{0,\infty})$. This proves immediately that $S-T_\infty$ is not an open 3-cell, for in an open 3-cell any compact set R is contained in a closed 3-cell J whose complement is simply connected.

References

1. L. Antoine, C. R. Acad. Sci. Paris vol. 171 (1920) pp. 661–663.

1a. ———, J. Math. Pures Appl. (8) vol. 4 (1921) pp. 221–325.

2. J. W. Alexander, Proc. Nat. Acad. Sci. U.S.A. vol. 10 (1924) pp. 8–12.

3. W. A. Blankinship, Thesis, Princeton University, 1949.

4. R. H. Fox and Emil Artin, Ann. of Math. vol. 49 (1948) pp. 979–990.

5. M. H. A. Newman and J. H. C. Whitehead, Quart. J. Math. Oxford Ser. vol. 8 (1937) pp. 14–21.

6. R. P. Coelho, Portugaliae Mathematica vol. 6 (1947) pp. 57–65.

7. H. Seifert and W. Threlfall, *Lehrbuch der Topologie*.

8. A. Kurosh, C. R. (Doklady) Acad. Sci. URSS. N.S. vol. 24 (1939) pp. 99–101, or *Teoriya grupp*, Moscow-Leningrad, 1944, chap. 10, §48.

PRINCETON UNIVERSITY

SOME LIMITING DISTRIBUTIONS RELATED TO THE SUM OF A RANDOM NUMBER OF RANDOM VARIABLES[1]

EDWARD PAULSON

1. Introduction. Let X_j $(j=1, 2, \cdots)$ be a sequence of independent random variables with the same distribution, and consider the random variable $Y=X_1+X_2+\cdots+X_N$. The limiting distribution of Y (as $\lambda\to\infty$) when N is an integer-valued random variable whose distribution depends on a parameter λ has been studied in detail by Robbins [1].[2] In this paper we shall attempt to extend some of Robbins' results by considering a more general type of statistic Y_1, which reduces to Y in a special case. We shall also consider the limiting distribution of a second statistic Y_2, which reduces to the arithmetic mean Y/N in a special case.

Where possible, we shall follow Robbins' notation. The probability that $N=k$ $(k=0, 1, 2, \cdots)$ will be denoted by ω_k, where the $\omega_k=\omega_k(\lambda)$ are functions of λ so that, for all λ, $\omega_k\geqq 0$ and $\sum_{k=0}^{\infty}\omega_k=1$. Furthermore, we set

$$\alpha = E(N) = \sum_{k=0}^{\infty}\omega_k\cdot k,$$

$$y^2 = \operatorname{Var}(N) = \sum_{k=0}^{\infty}\omega_k\cdot(k-\alpha)^2 \qquad (y^2<\infty \text{ for all } \lambda),$$

$$\theta(t) = E\left[e^{it((N-\alpha)/y)}\right] = \sum_{k=0}^{\infty}\omega_k\cdot e^{it((k-\alpha)/y)}.$$

Clearly α, y^2, and $\theta(t)$ are functions of λ. We restrict ourselves in this paper to cases for which

$$\lim_{\lambda\to\infty}\theta(t)=g(t)$$

where $g(t)$ is a characteristic function, so that $(N-\alpha)/y$ has a limiting distribution as $\lambda\to\infty$.

2. The statistic Y_1. The statistic Y_1 to be considered is to have a distribution function $F(u)=P\{Y_1\leqq u\}$ of the form

$$F(u) = \sum_{k=0}^{\infty}\omega_k\cdot F_k(u)$$

where $F_k(u)$ denotes the conditional distribution of Y_1 when $N=k$;

Received by the editors April 15, 1949 and, in revised form, July 23, 1949.

[1] Work done under the sponsorship of the Office of Naval Research.

[2] Numbers in brackets refer to the references cited at the end of the paper.

this conditional distribution $F_k(u)$ shall be subject to the essential restriction that there exist constants a and c (with $c>0$) so that

$$\lim_{k\to\infty}\int_{-\infty}^{\infty} e^{it((u-ak)/ck^{1/2})}dF_k(u) = q(t) \tag{1}$$

where $q(t)$ is another characteristic function. It is clear that Y_1 will reduce to Y when $F_k(u)$ is the kth convolution with itself of a random variable X such that $E(X)=a$, $E(X-a)^2=c^2$, $0<c^2<\infty$.

Now let $\sigma^2=\alpha c^2+y^2a^2$, and consider the random variable

$$Z_1 = \frac{Y_1 - a\alpha}{\sigma}.$$

THEOREM 1. *Suppose $a\neq 0$. If as $\lambda\to\infty$ limit $\alpha=\infty$, limit $\alpha/y^2=m$ $(0<m<\infty)$, then Z_1 has a limiting distribution whose characteristic function $\Phi(t)$ is given by*

$$\Phi(t) = g\left(t\left(\frac{a^2}{a^2+c^2m}\right)^{1/2}\right)\cdot q\left(t\left(\frac{c^2m}{a^2+c^2m}\right)^{1/2}\right).$$

The proof consists in showing that $\lim_{\lambda\to\infty} E(e^{itZ_1})=\Phi(t)$ for every real t. In the proof we shall use $o(1)$ to denote a quantity which $\to 0$ as $\lambda\to\infty$, and $o^*(1)$ to denote a quantity which $\to 0$ as $k\to\infty$.

We have

$$E(e^{itZ_1}) = \sum_{k=0}^{\infty}\omega_k\cdot\int_{-\infty}^{\infty} e^{it((u-a\alpha)/\sigma)}dF_k(u).$$

Since the hypotheses of Theorem 1 imply $y\to\infty$ as $\lambda\to\infty$, we find that $P\{|(N-\alpha)/y|>y^{1/2}\}=o(1)$ from Tchebycheff's inequality. Hence

$$\begin{aligned} E(e^{itZ_1}) &= o(1) + \sum_{|k-\alpha|\leq y^{3/2}}\omega_k\cdot\int_{-\infty}^{\infty} e^{it((u-a\alpha)/\sigma)}dF_k(u) \\ &= o(1) + \sum_{|k-\alpha|\leq y^{3/2}}\omega_k e^{ia t((k-\alpha)/\sigma)}\int_{-\infty}^{\infty} e^{it'((u-ak)/ck^{1/2})}dF_k(u) \end{aligned}$$

$$\text{where}\quad t' = t\left(\frac{ck^{1/2}}{\sigma}\right).$$

Since $t(ck^{1/2}/\sigma)=t(c^2m/(c^2m+a^2))^{1/2}+o(1)$ for $|k-\alpha|\leq y^{3/2}$, we see that t' lies in a finite interval for any fixed value of t. Since (1) implies uniform convergence in any finite interval, we have uniformly for all k such that $|k-\alpha|\leq y^{3/2}$

$$\int_{-\infty}^{\infty} e^{it'((u-ak)/c\lambda^{1/2})} dF_k(u) = q(t') + o^*(1)$$

$$= q(t') + o(1)$$

$$= q\left(t\left(\frac{c^2m}{c^2m + a^2}\right)^{1/2}\right) + o(1),$$

making use of the fact that $\alpha - y^{3/2} \to \infty$ as $\lambda \to \infty$, and that a characteristic function is everywhere continuous. Combining these results we obtain

$$E(e^{itZ_1}) = o(1) + q\left(t\left(\frac{c^2m}{c^2m + a^2}\right)^{1/2}\right) \sum_{|k-\alpha| \leqq y^{3/2}} \omega_k \cdot e^{iat((k-\alpha)/\sigma)}.$$

Now

$$\sum_{|k-\alpha| \leqq y^{3/2}} \omega_k \cdot e^{iat((k-\alpha)/\sigma)} = o(1) + \sum_{k=0}^{\infty} \omega_k \cdot e^{iat(y/\sigma)((k-\alpha)/y)}$$

$$= o(1) + \left[o(1) + g\left(t\frac{ay}{\sigma}\right)\right]$$

$$= o(1) + g\left(t\left(\frac{a^2}{c^2m + a^2}\right)^{1/2}\right).$$

We now have

$$E(e^{itZ_1}) = o(1) + g\left(t\left(\frac{a^2}{c^2m + a^2}\right)^{1/2}\right) \cdot q\left(t\left(\frac{c^2m}{c^2m + a^2}\right)^{1/2}\right),$$

which proves Theorem 1.

By proceeding in a similar manner, it is easy to verify that if either of the two sets of conditions (1) $a \neq 0$, limit $\alpha = \infty$, limit $\alpha/y^2 = \infty$, or (2) $a = 0$, limit $\alpha = \infty$, limit $\alpha/y^2 = m'$ $(0 < m' \leqq \infty)$ hold, then Z_1 has a limiting distribution with characteristic function $q(t)$.

3. **The statistic** Y_2. The statistic Y_2 again has a distribution function $F(u) = P\{Y_2 \leqq u\}$ of the form

$$F(u) = \sum_{k=0}^{\infty} \omega_k \cdot F_k(u)$$

but the conditional distribution $F_k(u)$ of Y_2 when $N = k$ is now subject to the restriction that there exist a constant a such that

$$\lim_{k \to \infty} \int_{-\infty}^{\infty} e^{iuk^{1/2}(u-a)} dF_k(u) = m(t)$$

where $m(t)$ is a characteristic function.

Now let $Z_2=\alpha^{1/2}(Y_2-a)$.

THEOREM 2. *If as* $\lambda\to\infty$ *limit* $\alpha=\infty$, *limit* $\alpha/y^2=m'$ $(0<m'\leqq\infty)$, *then* Z_2 *has a limiting distribution corresponding to the characteristic function* $m(t)$.

The proof again consists in showing that limit$_{\lambda\to\infty}$ $E(e^{itZ_2})=m(t)$.

$$
\begin{aligned}
E(e^{itZ_2}) &= \sum_{k=0}^{\infty} \omega_k \cdot \int_{-\infty}^{\infty} e^{it\alpha^{1/2}(u-a)}dF_k(u) \\
&= o(1) + \sum_{|k-\alpha|\leq y^{3/2}} \omega_k \cdot \int_{-\infty}^{\infty} e^{it(\alpha/k)^{1/2}[u^{1/2}(u-a)]}dF_k(u) \\
&= o(1) + \sum_{|k-\alpha|\leq y^{3/2}} \omega_k \left[m\left(t\left(\frac{\alpha}{k}\right)^{1/2}\right) + o(1)\right] \\
&= o(1) + m(t) \sum_{|k-\alpha|\leq y^{3/2}} \omega_k \\
&= o(1) + m(t),
\end{aligned}
$$

which proves Theorem 2.

4. **Some examples.** We conclude this paper by giving an example involving Theorem 1, and an example involving Theorem 2. First let N have a Poisson distribution with mean λ, so that $\omega_k=e^{-\lambda}\lambda^k/k!$, $\alpha=\lambda$, $y^2=\lambda$. Let $\bar{x}=(1/N)\sum_{i=1}^{N} x_i$ and put $Y_1=\sum_{i=1}^{N}(x_i-\bar{x})^2$ where $\{x_i\}$ $(i=1, 2, \cdots)$ is a sequence of independent observations on a random variable whose fourth central moment μ_4 is finite. If we make use of the known fact that the sample variance (based on k observations) is asymptotically normal with mean μ_2 and variance $(\mu_4-\mu_2^2)/k$, it follows from Theorem 1 that as $\lambda\to\infty$ the limiting distribution of $(Y_1-\lambda\mu_2)/(\lambda\mu_4)^{1/2}$ is normal with zero mean and unit variance.

For the second example, let N have a binomial distribution with parameters λ, p, $q=1-p$, so that $\omega_k=C_{\lambda,k}p^kq^{\lambda-k}$ $(k=0, 1, 2, \cdots, \lambda)$. Now let $Y_2=r/N$ where r is the number of successes in N independent trials with constant probability P of a success. It follows from Theorem 2 that $(\lambda p)^{1/2}[Y_2-P]$ has in the limit a normal distribution with zero mean and variance $P(1-P)$. The statistic in the second example arose in a problem considered by Birnbaum and Sirken where Y_2 represented the percentage of people voting yes when only N out of a random sample of λ were available for interviewing. The limiting distribution of Y_2 for this special case was obtained by Birnbaum and Sirken by a different method.

References

1. H. Robbins, *The asymptotic distribution of the sum of a random number of random variables*, Bull. Amer. Math. Soc. vol. 54 (1948) pp. 1151–1162.
2. H. Cramér, *Mathematical methods of statistics*, Princeton University Press, 1946, p. 365.
3. Z. W. Birnbaum and M. Sirken, *Bias due to non-availability in sampling surveys*, Journal of the American Statistical Association vol. 45 (1950) pp. 98–111.

UNIVERSITY OF MICHIGAN

BOUNDS FOR THE COEFFICIENTS OF UNIVALENT FUNCTIONS

ARYEH DVORETZKY

1. The purpose of this note is to give some estimates for the moduli of the coefficients a_n of

$$w = f(z) = z + a_2z^2 + \cdots + a_nz^n + \cdots, \tag{1}$$

assumed regular and univalent in $|z| < 1$, in terms of the domain onto which $|z| < 1$ is mapped through (1). A typical result, cf. (27), is that if this domain does not cover arbitrarily large circles, then[1] $a_n = O(\log n)$.

Let W be the domain in the w-plane onto which $|z| < 1$ is mapped through (1) and denote by $A(R)$ the radius of the largest circle with center on $|w| = R$ the whole interior of which is contained in W, that is,

$$A(R) = \max_{|w|=R} \min_{w' \notin W} |w' - w| \qquad (0 \leqq R < \infty).$$

Our aim is to derive upper bounds for $|a_n|$ in terms of $A(R)$.

Always $A(R) \leqq R+1$ while $A(R) = 0$ for sufficiently large R if and only if (1) is bounded in $|z| < 1$. The function $A(R)$ measures, in some sense, the extension of W. This "extension" has, however, little to do with area, in fact whatever the positive function $a(R)$ there exists a function (1) mapping $|z| < 1$ on a slit domain and for which $A(R) = o(a(R))$.

Presented to the Society, August 30, 1949; received by the editors March 31, 1949 and, in revised form, July 6, 1949.

[1] Throughout the paper, O and o refer to $n \to \infty$ or $R \to \infty$ according to which of the variables n or R appears in the formula.

It is well known that $a_n = O(n)$ for all functions (1) while $a_n = o(n^{-1/2})$ for bounded ones. These correspond respectively to no restrictions on $A(R)$ and to $A(R) = 0$ for large R. As an application of our results we shall find a gradual transition between these extreme cases (cf. [1][2] where some of the results proved here were announced).

Very probably the estimates obtained here do not reflect in most cases even the correct order of magnitude. At any rate it seems that their main interest is in the qualitative, rather than the quantitative, aspect. Therefore we did not push as much as possible the evaluation of the constants involved, and also refrained from giving some rather lengthy refinements (similar to those used by Landau [2] to improve Littlewood's $|a_n| < en$) which would slightly strengthen our results.

2. It is well known that for $|z| \leqq t < 1$, (1) satisfies

$$|f(z)| \leqq M(t) = \frac{t}{(1-t)^2}. \tag{2}$$

If $\beta \notin W$, then

$$g(z) = \frac{f(z)}{1 - f(z)/\beta} = z + \cdots$$

is also univalent in $|z| < 1$. Writing M for $M(t)$ and applying (2) to $g(z)$, we see that if

$$|\beta| < M \tag{3}$$

holds, then (1) maps $|z| < t$ on a domain W_t exterior to the circle

$$\left| w - \frac{M^2\beta}{M^2 - |\beta|^2} \right| = \frac{M|\beta|^2}{M^2 - |\beta|^2}. \tag{4}$$

Now let θ be real, B positive, and b satisfy

$$0 < b < 1, \tag{5}$$

and assume

$$0 \leqq R \leqq bM \tag{6}$$

and

$$|\beta - Re^{i\theta}| < B\frac{R^2}{M}. \tag{7}$$

[2] Numbers in brackets refer to the bibliography at the end of the paper.

Then, if (3) holds, which certainly is the case if

$$B \leqq \frac{1-b}{b^2} \tag{8}$$

is satisfied, we have

$$\begin{aligned}\left|\frac{M^2Re^{i\theta}}{M^2-R^2}-\frac{M^2\beta}{M^2-|\beta|^2}\right| &= \frac{M^2}{M^2-|\beta|^2}\left|\frac{M^2-|\beta|^2}{M^2-R^2}Re^{i\theta}-\beta\right| \\ &= \frac{M^2}{M^2-|\beta|^2}\left|Re^{i\theta}-\beta+\frac{R^2-|\beta|^2}{M^2-R^2}Re^{i\theta}\right| \\ &\leqq \frac{M^2}{M^2-|\beta|^2}|Re^{i\theta}-\beta|\left(1+\frac{R+|\beta|}{M^2-R^2}R\right).\end{aligned}$$

Thus, if

$$M|Re^{i\theta}-\beta|(M^2+R|\beta|) \leqq |\beta|^2(M^2-R^2) \tag{9}$$

holds, then by (4)

$$\frac{M^2}{M^2-R^2}Re^{i\theta} \notin W_t. \tag{10}$$

We note that, because of (6) and (7), (9) is implied by $bB<1$ and

$$(1+b^2+b^3B)B \leqq (1-b^2)(1-bB)^2. \tag{11}$$

The quadratic equation in B obtained from (11) on replacing the inequality by an equality sign has a unique root in the interval $0<B<1$. Let $B(b)$ be this root. Clearly $B(b)\to 0$ as $b\to 1$ and $B(b)\to 1$ as $b\to 0$.

It is easily verified that $B=B(b)$ satisfies (8). We have thus from (10) that if

$$A(R) < B(b)\frac{R^2}{M}, \tag{12}$$

then no point of $|w|=M^2R/(M^2-R^2)$ is contained in W_t and thus

$$W_t \subset \frac{M^2R}{M^2-R^2}.$$

Or, in other words, (5), (6), *and* (12) *imply*

$$|f(z)| < \frac{M^2R}{M^2 - R^2} \leqq \frac{R}{1 - b^2} \tag{13}$$

for $|z| \leqq t$. (t enters through (6) and (13) since $M = M(t)$.)

3. Before applying (13) we go through a reasoning devised by Littlewood [3] and somewhat simplified by Landau [2].

With (1) also

$$\phi(z) = (f(z^2))^{1/2} = \sum_{n=1}^{\infty} \alpha_n z^n \qquad (\alpha_1 = 1) \tag{14}$$

is regular and univalent in $|z| < 1$. If we put

$$H(t) = \max_{|z|=t} \left| \frac{f(z)}{z} \right| \qquad (0 \leqq t < 1), \tag{15}$$

then

$$|\phi(z)| \leqq (tH(t))^{1/2} \qquad \text{for } |z| \leqq t^{1/2} < 1.$$

Hence, since (14) is univalent, by evaluating the area

$$\sum_{n=1}^{\infty} n |\alpha_n|^2 t^n \leqq tH(t)$$

and on dividing by t and integrating, we get

$$\sum_{n=1}^{\infty} |\alpha_n|^2 r^n \leqq \int_0^r H(t)dt \qquad (0 \leqq r < 1). \tag{16}$$

But from (14) we have

$$|a_n| r^n = \left| \sum_{\nu=1}^{2n-1} \alpha_\nu r^{\nu/2} \alpha_{2n-\nu} r^{n-\nu/2} \right| \leqq \frac{1}{2} \sum_{\nu=1}^{2n-1} (|\alpha_\nu|^2 r^\nu + |\alpha_{2n-\nu}|^2 r^{2n-\nu})$$

$$= \sum_{\nu=1}^{2n-1} |\alpha_\nu|^2 r^\nu;$$

thus, by (16), we have for every $0 < r < 1$

$$|a_n| \leqq \frac{1}{r^n} \int_0^r H(t)dt. \tag{17}$$

(Equality can occur only for $n = 1$ when $f(z)$ reduces to z.)

4. As a first application we prove the following:

There exists a finite positive K such that

$$\limsup_{n=\infty} \frac{|a_n|}{n} \leqq KL \tag{18}$$

holds for all functions (1) *for which*

$$\limsup_{R=\infty} \frac{A(R)}{R} \leqq L. \tag{19}$$

PROOF. Let b_0 be any value of b satisfying (5), put $B_0=B(b_0)$, and choose L_0 with

$$0 < L_0 < b_0 B_0. \tag{20}$$

$A(R)$ is a continuous function of R satisfying $A(0)>0$ and $A(R) \leqq R+1$. Therefore for every $0<t<1$ the equation in R

$$A(R) = B_0 R^2 \frac{(1-t)^2}{t} \tag{21}$$

has positive finite solutions. These solutions clearly form a closed set and we denote by $R_t=R_t(f, b_0)$ the smallest among them. From (13) we have:

If

$$R_t \leqq \frac{b_0 t}{(1-t)^2}, \tag{22}$$

then

$$\max_{|z| \leqq t} |f(z)| \leqq \frac{R_t}{1-b_0^2}. \tag{23}$$

Note that if $A(R)$ is replaced in (21) by a larger function, the effect is to increase R_t.

Let now L satisfy $0 \leqq L < L_0$ and take L' in the interval $L<L'<L_0$. Then, using the remark just made, we have for large R (thus necessarily for $t>t_0$ with suitably fixed $t_0<1$)

$$R_t < \frac{L't}{B_0(1-t)^2}. \tag{24}$$

Because of (20), (22)—and thus also (23)—is satisfied. From (17), (15), (2), (23), and (24) we obtain

$$|a_n| r^n \leqq \int_0^{t_0} \frac{dt}{(1-t)^2} + \frac{L'}{B_0(1-b_0^2)} \int_0^r \frac{dt}{(1-t)^2};$$

integrating and taking $r=1-1/n$ we have

$$|a_n| < O(1) + \frac{L'en}{B_0(1-b_0^2)},$$

or, since L' may be taken as near L as we please,

$$\limsup_{n=\infty} \frac{|a_n|}{n} \leqq \frac{Le}{B_0(1-b_0^2)}.$$

On the other hand we always have according to Littlewood [2] (by an application of (2) and (17) with $r=1-1/n$) $|a_n| < en$. Therefore

$$K = \max\left(\frac{e}{B_0(1-b_0^2)}, \frac{e}{L_0}\right)$$

satisfies (18) as required, q.e.d.

5. Next we proceed to show how our results allow the gradual passage from $a_n=O(n)$ in the general case to $a_n=o(n^{-1/2})$ in the bounded one:

Let

$$A(R) = O(R^\gamma), \tag{25}$$

then we have

$$a_n = O(n^{\gamma/(2-\gamma)}) \quad \text{for } 0 < \gamma \leqq 1, \tag{26}$$

$$a_n = O(\log n) \quad \text{for } \gamma = 0, \tag{27}$$

$$a_n = O(1) \quad \text{for } -2 \leqq \gamma < 0, \tag{28}$$

$$a_n = O(n^{-1/2+2/(2-\gamma)}) \quad \text{for } \gamma < -2. \tag{29}$$

PROOF. $\gamma=1$ need not be considered and in all other cases R_t automatically satisfies (22) (indeed a stronger statement was established in §4).

From (25) and (21) we have for $t>t_0$, with suitable $t_0<1$ and corresponding K_0,

$$R_t < K_0(1-t)^{2/(\gamma-2)}. \tag{30}$$

Applying (17), with $H(t)$ estimated from (2) for $t \leqq t_0$, and from (30) in $t_0<t<1$ we get (with constant $K_1, \cdots, K_5$)

$$|a_n| r^n < O(1) + K_1 \int_0^r (1-t)^{2/(\gamma-2)} dt.$$

Thus if $0<\gamma<2$ we have

$$|a_n| r^n < K_2(1-r)^{\gamma/(\gamma-2)}$$

while for $\gamma=0$ we get

$$|a_n| r^n < K_2 \log \frac{1}{1-r}.$$

For $r=1-1/n$ these formulas yield (26) and (27).

Similarly for $\gamma<0$, we have $|a_n| r^n<K_3$ and thus (28).

To obtain (29), we estimate from (13) and (30) the area of W_t, thus obtaining (for $t>t_0$)

$$\sum n |a_n|^2 t^{2n} < K_4 R_t^2 < K_5(1-t)^{4/(\gamma-2)}.$$

This, for $t=1-1/n$, yields

$$n|a_n|^2 = O(n^{4/(2-\gamma)}),$$

and thus (29).

6. As remarked at the beginning of the paper, the methods used can be manipulated to give somewhat better results. The O results of the last section can easily be replaced by results involving constants. In particular if, in the assumption (25), O is replaced by o, the same may be done in the conclusions (26), (27), and (29).

Also instead of conditions such as (25) referring to $A(R)$ for all R sufficiently large, it is possible to use conditions on mean values of $A(R)$. Thus (28) holds if $\int_0^1 R_t dt$ is finite, and this can be secured by suitable assumptions on the mean value of $A(R)$.

Evidently, instead of R^γ in (25) we could equally well use other functions, for example e^{R^γ}, and so on.

Finally let it be remarked that the methods of this paper may also be applied to other classes of functions, in particular to functions which are p-valent (or p-valent in the mean).

Bibliography

1. A. Dvoretzky, *Les coefficients d'une fonction univalente et le domain étalé*, C.R. Acad. Sci. Paris vol. 225 (1947) pp. 447–449.

2. E. Landau, *Über schlichte Funktionen*, Math. Zeit. vol. 30 (1929) pp. 635–638.

3. J. E. Littlewood, *On inequalities in the theory of functions*, Proc. London Math. Soc. (2) vol. 23 (1925) pp. 481–519.

Hebrew University and
Institute for Advanced Study

LIMITING VALUES OF SUBHARMONIC FUNCTIONS[1]

ELMER TOLSTED

1. **Introduction.** In 1928, Littlewood obtained the following result [1, p. 393].[2]

(1.1) THEOREM. *Let $u(r, \theta)$ be subharmonic in the unit circle, $r<1$, and satisfy*

$$\int_{-\pi}^{\pi} |u(r, \theta)|\, d\theta = O(1), \qquad r < 1. \tag{1.2}$$

Then there exists a finite-valued function $U(\theta)$, $0 \leqq \theta < 2\pi$, such that $\lim_{r \to 1} u(r, \theta) = U(\theta)$ for almost all values of θ.

In 1934, Priwaloff published a generalization of Littlewood's result, which turned out to be incorrect. When the domain under consideration is the unit circle, then Priwaloff's generalization consisted in allowing "non-tangential" approaches to the boundary of the disc.

In 1942, during the course of his lectures on *Subharmonic functions* at Brown University, the late Professor J. D. Tamarkin discovered an error in Priwaloff's proof. Then in a letter to Tamarkin in 1943, Professor A. Zygmund described a counter-example to Priwaloff's result.

In this paper, we present several generalizations of Littlewood's result (see §3) as well as several counter-examples to Priwaloff's result (see §4).

2. **Definitions and lemmas.** Points on the unit circle will be denoted by $e^{i\xi}$; the point (1, 0) will be denoted by B. The coordinates (x, y) and (r, θ) will be used interchangeably to denote the point P, and (ρ, ϕ) will be used to denote the point Q; here P and Q are points in the unit disc. The point inverse to P with respect to the unit circle will be denoted by $P'(1/r, \theta)$.

The letter A denotes a positive absolute constant, though not always the same from one occurance to another.

We shall use the notation $s = 1 - r$ and $\sigma = 1 - \rho$.

The symbol $L_\psi(\xi)$ will denote the (short) line segment approaching the point $e^{i\xi}$ from the interior of the unit circle and making an angle ψ

Presented to the Society, November 29, 1947; received by the editors May 26, 1949.

[1] Part of a thesis submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy at Brown University.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

with the radius vector to the point $e^{i\xi}$. We shall consider ψ positive if the rotation from the radius vector to $L_\psi(\xi)$ is clockwise. For a fixed angle ψ, $|\psi| < \pi/2$, the family of lines $L_\psi(\xi)$, $0 \leqq \xi < 2\pi$, has for its envelope a circle of radius $\rho_\psi = |\sin \psi|$ inside and concentric with the unit circle. We shall let $S_\psi(\xi)$, $0 \leqq \psi < \pi/2$, denote the area in the ring $\rho_\psi \leqq \rho < 1$ between the lines $L_\psi(\xi)$ and $L_{-\psi}(\xi)$. We shall use $\bar{S}_\psi(\xi)$ to denote the area in the ring $\rho_\psi \leqq \rho < 1$ which is outside the sector $S_\psi(\xi)$.

The phrase "P may approach the unit circumference at the point $e^{i\xi}$ along any non-tangential path" is equivalent to the statement "P may approach the point $e^{i\xi}$ along any path which remains within some sector $S_\psi(\xi)$, $\psi < \pi/2$." For brevity this type of approach shall be designated by $P \to_{S_\psi(\xi)} e^{i\xi}$. In general the nature of the path of approach shall be indicated as a subscript to the symbol $\to$.

A simple geometric argument yields the following result.

(2.1) LEMMA. *Let $P(r, \theta)$ be a point on the segment $L_\alpha(0)$, $\alpha < \pi/2$, and let $Q(\rho, \phi)$ be a point on the segment $L_\alpha(\tau)$. Then there exists a constant k_1, $0 < k_1 < \infty$, such that $r > \rho_\alpha + (1-\rho_\alpha)/2$ implies $PQ \geqq k_1\tau$, $0 \leqq \tau < 2\pi$.*

The Green's function for the unit disc is given by

$$g(P, Q) = \log \frac{rQP'}{QP}.$$

Littlewood showed that

$$0 \leqq g(P, Q) \leqq A\sigma s/PQ^2 \tag{2.2}$$

holds [1, p. 394]. We shall require further estimates of $g(P, Q)$, similar to (2.2). To that end let α be fixed, $|\alpha| < \pi/2$; then we can locate the point $Q(\rho, \phi)$ by the coordinates (σ, τ) where $\sigma = 1-\rho$, and $L_\alpha(\tau)$ is the segment passing through Q meeting the unit circumference at $e^{i\tau}$. We shall divide the ring $\rho > \rho_0$ into three sets R_1, R_2, and R_3 defined explicitly in terms of the coordinates σ and τ of the point Q as follows:

$$\begin{aligned} R_1&: \quad |\sigma - s| \leqq s/2, \quad |\tau| \leqq s, \quad \rho > \rho_0, \\ R_2&: \quad |\sigma - s| > s/2, \quad |\tau| \leqq s, \quad \rho > \rho_0, \\ R_3&: \quad |\tau| > s, \quad \rho > \rho_0. \end{aligned}$$

In R_1 we shall prove

$$g(P, Q) \leqq \frac{\sigma}{s} \log \frac{As^2}{\tau^2}, \quad \text{for} \quad s < \frac{1}{2}. \tag{2.3}$$

We know that

$$g(P, Q) = \log \frac{r \cdot P'Q}{PQ} = \frac{1}{2} \log \frac{r^2 \cdot P'Q^2}{PQ^2}. \tag{2.4}$$

First note that $r<1$, and for Q in R_1 we have $PQ \leqq As$. Also $PP' = 1/r - r = (1-r)(1+r)/r \leqq 4s$ for $s<1/2$, and $P'Q \leqq PP' + PQ \leqq As$. At this point we use (2.1) to obtain the important fact that $PQ \geqq k_1 \tau$ where $0<k_1<\infty$. Thus

$$g(P, Q) \leqq \frac{1}{2} \log \frac{As^2}{k_1^2 \tau^2}$$

and (2.3) follows at once from the fact $|\sigma - s| \leqq s/2$ implies $1/2 \leqq \sigma/s \leqq 3/2$.

In R_2 we shall prove

$$g(P, Q) \leqq \frac{A\sigma s}{s^2 + \tau^2}. \tag{2.5}$$

By (2.2) we have that

$$g(P, Q) \leqq \frac{A\sigma s}{PQ^2} = \frac{A\sigma s}{PQ^2/2 + PQ^2/2}.$$

But for Q in R_2, $|\sigma - s| > s/2$; and hence $PQ^2/2 \geqq s^2/8$. By (2.1), $PQ^2/2 \geqq k_1 \tau^2/2$, and so we have

$$g(P, Q) \leqq \frac{A\sigma s}{s^2/8 + k_1^2 \tau^2/2}$$

which implies (2.5).

In R_3 we have

$$g(P, Q) \leqq \frac{A\sigma s}{\tau^2} \tag{2.6}$$

which follows at once from (2.2) and (2.1).

Finally, we shall need the following simple results on approximations [1, §3].

Suppose that $J^*(\tau)$ is non-negative and monotone increasing to a finite $J^*(\pi)$ in $0 \leqq \tau \leqq \pi$ and satisfies

$$\limsup_{\tau \to +0} \frac{J^*(\tau)}{\tau} \leqq \eta(\rho_0);$$

then

$$\limsup_{s \to +0} s \int_{s-0}^{\pi} \frac{dJ^*(\tau)}{\tau^2} \leqq A\eta(\rho_0), \tag{2.7}$$

$$\limsup_{s \to +0} s \int_{0}^{s} \frac{dJ^*(\tau)}{s^2 + \tau^2} \leqq A\eta(\rho_0), \tag{2.8}$$

and

$$\limsup_{s \to +0} \frac{1}{s} \int_{0}^{s} \log \frac{As}{\tau^2}\, dJ^*(\tau) \leqq A\eta(\rho_0). \tag{2.9}$$

3. **Generalizations of Littlewood's theorem.** If $u(P)$ is subharmonic in the unit circle, and if $u(P)$ satisfies (1.2), then $u(P)$ can be represented as

$$u(P) = h(P) - \iint g(P, Q)dF(Q) \tag{3.1}$$

where $h(P)$ is harmonic in the circle $r<1$, and satisfies

$$\int_{-\pi}^{\pi} |\, h(r, \theta)\,|\, d\theta = O(1), \qquad r < 1, \tag{3.2}$$

and $F(Q)$, the generalized mass distribution function associated with $u(P)$, is a non-negative, additive function of sets satisfying

$$\iint (1 - \rho)dF(Q) < +\infty. \tag{3.3}$$

The integrals in (3.1) and (3.3) are Stieltjes-Radon integrals extended over the unit circle, $\rho<1$ [1, Lemma 3].

Since $h(P)$ satisfies condition (3.2), the limit of $h(P)$ exists for almost all values of ξ as P approaches $e^{i\xi}$ along any non-tangential path (see [4, Chapter IV, Theorem 1 and Chapter II, Theorem 7]). Having thus disposed of the harmonic part of (3.1), we have reduced our problem to the consideration of the behavior of the potential of positive mass distribution

$$w(P) = \iint g(P, Q)dF(Q) \tag{3.4}$$

as P approaches the unit circumference.

Littlewood showed that $w(P)$ had radial limit zero at almost all points of the unit circumference [1, Lemma 4]. Priwaloff tried unsuccessfully to show that $w(P)$ had a non-tangential limit zero at almost all points of the unit circumference. Since Littlewood showed

that (1.2) and the combination of (3.1) and (3.3) are equivalent restrictions on the subharmonic function $u(P)$, the results in this paper are derived from the consideration of potential functions of the type (3.4) whose associated mass distribution functions $F(Q)$ satisfy (3.3). Such potential functions of positive mass distribution are superharmonic (see [3, 4.34] and [3, 1.1]). Since the negative of a superharmonic function is subharmonic, we may apply the theory of subharmonic functions, with obvious modifications, to our potential functions (3.4) whenever necessary.

We shall use the method of Littlewood to generalize (1.1) from the case of radial approach to the case in which the approach to the boundary of the unit circle is along the rotations of a fixed line segment or a fixed curve.

(3.5) THEOREM. *Let $u(P)$ be subharmonic in the unit circle, $r<1$, and satisfy (1.2). Let $L_\alpha(\xi)$ be a straight line segment which approaches the unit circumference at the point $e^{i\xi}$ making a fixed angle $\alpha<\pi/2$ with the radius vector to the point $e^{i\xi}$. Then there exists a finite valued function $U(\xi)$, $-\pi\leqq\xi<\pi$, such that, for almost all values of ξ,*

$$\lim_{P\to L_\alpha(\xi)e^{i\xi}} u(P) = U(\xi).$$

PROOF. In view of the preceding remarks it will be sufficient to prove with the aid of (3.3) that

$$\lim_{P\to L_\alpha(\xi)e^{i\xi}} \iint g(P, Q)dF(Q) = 0 \tag{3.6}$$

for almost all values of ξ, $0\leqq\xi<2\pi$.

Let

$$\epsilon(\rho_0) = \iint_{\rho>\rho_0} (1-\rho)dF(Q) \tag{3.7}$$

and let $\eta(\rho_0)=(\epsilon(\rho_0))^{1/2}$. By (3.3) we have that $\lim_{\rho_0\to 1}\epsilon(\rho_0)=0$ and thus that $\lim_{\rho_0\to 1}\eta(\rho_0)=0$.

Next we shall define

$$\Phi(\xi) = \iint \sigma dF(Q) \tag{3.8}$$

over the part of the ring $\rho>\rho_0$ between and including the segments $L_\alpha(0)$ and $L_\alpha(\xi)$. From the properties of σ, $F(Q)$, and $L_\alpha(\xi)$ it follows that for ρ_0 near to one, $\Phi(\xi)$ is a monotone increasing function of ξ. In view of (3.7), $\Phi(\xi)$ remains less than $2\epsilon(\rho_0)$, $0\leqq\xi\leqq 2\pi$, and $\Phi'(\xi)$ exists for almost all values of ξ.

Let $E(\rho_0)$ be the set of values of ξ, $0 \leqq \xi < 2\pi$, for which

$$0 \leqq \Phi'(\xi) \leqq 2\eta(\rho_0) \tag{3.9}$$

and let C.E. be the complement of E. From the relations

$$2\eta^2(\rho_0) = 2\epsilon(\rho_0) \geqq \Phi(2\pi) \geqq \int_0^{2\pi} \Phi'(\xi)d\xi$$

$$\geqq \int_{C.E.} \Phi'(\xi)d\xi \geqq m(C.E.)2\eta(\rho_0)$$

we may conclude that the measure of C.E. does not exceed $\eta(\rho_0)$.

At this set $E(\rho_0)$ whose complement C.E. has measure not exceeding $\eta(\rho_0)$ we shall show

$$\limsup_{P \to L_\alpha(\xi) e^{i\xi}} \iint_{\rho > \rho_0} g(P, Q)dF(Q) \leqq A\eta(\rho_0). \tag{3.10}$$

Since

$$\lim_{P \to e^{i\xi}} \iint_{\rho \leqq \rho_0} g(P, Q)dF(Q) = 0$$

uniformly in ξ, the left side of (3.10) is unaltered when the symbol $\rho > \rho_0$ is suppressed and the integration taken over the whole of $\rho < 1$. But then the left side of (3.10) does not depend on ρ_0; hence since $\eta(\rho_0)$ is arbitrarily small, (3.10) once proved implies (3.6) and establishes the theorem.

Let now

$$J(\xi, t) = \iint \sigma dF(Q) \tag{3.11}$$

over the portion of the ring $\rho > \rho_0$ between and including the segments $L_\alpha(\xi)$ and $L_\alpha(\xi + t)$. If now ξ is a point of $E(\rho_0)$ we have by the characteristic property of E

$$\limsup_{t \to 0} \frac{J(\xi, t)}{|t|} \leqq 2\eta(\rho_0). \tag{3.12}$$

We may assume without loss of generality that B, the point for which $\xi = 0$, is a point of the set $E(\rho_0)$. We then write $J(t)$ for $J(0, t)$ and define

$$J^*(t) = J(t) + |J(-t)|. \tag{3.13}$$

It follows from (3.12) and (3.13) that

$$\limsup_{t\to+0}\frac{J^*(t)}{t}\leqq 4\eta(\rho_0). \tag{3.14}$$

With the aid of (3.14) we shall prove (3.10) for the case $\xi=0$; namely

$$\limsup_{P\to L_\alpha(0)^B}\iint_{\rho>\rho_0} g(P,Q)dF(Q)\leqq A\eta(\rho_0), \tag{3.15}$$

thus establishing the theorem. In order to demonstrate (3.15) we divide the ring $\rho>\rho_0$ into the three regions R_1, R_2, and R_3 described in §2 and establish the theorem by showing that

$$\limsup_{P\to L_\alpha(0)^B}\iint_{R_i} g(P,Q)dF(Q)\leqq A\eta(\rho_0),\qquad i=1,2,3. \tag{3.16}$$

We shall need Fubini's theorem and the inequalities quoted in §2.

Let

$$\Lambda_1(P)=\iint_{R_1} g(P,Q)dF(Q). \tag{3.17}$$

By (2.3) we have

$$\Lambda_1(P)\leqq\iint_{R_1}\frac{\sigma}{s}\log\frac{As}{\tau^2}\,dF(\sigma,\tau);$$

and by (3.11), (3.13), and Fubini's theorem we may write

$$\Lambda_1(P)\leqq\int_0^s\frac{1}{s}\log\frac{As^2}{\tau^2}\,dJ^*(\tau).$$

But then by (2.9) and (3.14) we have that

$$\limsup_{s\to+0}\Lambda_1(P)\leqq\limsup_{s\to+0}\frac{1}{s}\int_0^s\log\frac{As^2}{\tau^2}\,dJ^*(\tau)\leqq A\eta(\rho_0);$$

and since $P\to B$ implies $s\to+0$, we have

$$\limsup_{P\to L_\alpha(0)^B}\Lambda_1(P)\leqq A\eta(\rho_0). \tag{3.18}$$

Let

$$\Lambda_2(P)=\iint_{R_2} g(P,Q)dF(Q). \tag{3.19}$$

By (2.5) we have

$$\Lambda_2(P) \leqq \iint_{R_2} \frac{A\sigma s}{s^2+\tau^2}\, dF(\sigma, \tau);$$

and by (3.11),(3.13), and Fubini's theorem we may write

$$\Lambda_2(P) \leqq \int_0^s \frac{As}{s^2+\tau^2}\, dJ^*(\tau).$$

But then by (2.8) and (3.14) we have that

$$\limsup_{s\to+0} \Lambda_2(P) \leqq \limsup_{s\to+0} As \int_0^s \frac{dJ^*(\tau)}{s^2+\tau^2} \leqq A\eta(\rho_0);$$

and since $P\to B$ implies $s\to+0$, we have

$$\limsup_{P\to L_\alpha(0)B} \Lambda_2(P) \leqq A\eta(\rho_0). \tag{3.20}$$

Let

$$\Lambda_3(P) = \iint_{R_3} g(P, Q)\, dF(Q). \tag{3.21}$$

By (2.6) we have

$$\Lambda_3(P) \leqq \iint_{R_3} \frac{A\sigma s}{\tau^2}\, dF(\sigma, \tau);$$

and again using (3.11), (3.13), and Fubini's theorem we may write

$$\Lambda_3(P) \leqq \int_s^\pi \frac{As}{\tau^2}\, dJ^*(\tau).$$

Then by (2.7) and (3.14) we have that

$$\limsup_{s\to+0} \Lambda_3(P) \leqq \limsup_{s\to+0} s \int_{s-0}^\pi \frac{A\,dJ^*(\tau)}{\tau^2} \leqq A\eta(\rho_0);$$

and since $P\to B$ implies $s\to+0$, we have finally

$$\limsup_{P\to L_\alpha(0)B} \Lambda_3(P) \leqq A\eta(\rho_0). \tag{3.22}$$

Combining the results (3.18), (3.20), and (3.22) gives us (3.16), thus proving the theorem.

(3.23) COROLLARY. *Let $C_\alpha(0)$ be a curve which approaches the unit circumference at B, and is tangent to $L_\alpha(0)$ at B. Let $C_\alpha(\xi)$ be the curve obtained by rotating the curve $C_\alpha(0)$ through an angle ξ about the origin. Then if $u(P)$ is the subharmonic function of Theorem* (3.5), *we have*

$$\limsup_{P \to C_\alpha(\xi)e^{i\xi}} u(P) = U(\xi)$$

for almost all values of ξ, $0 \leqq \xi \leqq 2\pi$.

PROOF. The proof goes through exactly as the proof of the theorem. The crucial part of the argument is the demonstration of Lemma (2.1) for P on $C_\alpha(0)$ and Q on $C_\alpha(\tau)$, which is needed to obtain the essential inequalities (2.3), (2.5), and (2.6). Lemma (2.1) may be proved for such curves $C_\alpha(0)$ and $C_\alpha(\tau)$ if they are tangent to the lines $L_\alpha(0)$ and $L_\alpha(\tau)$ respectively. If, however, the curve $C_\alpha(0)$ oscillated rapidly as it approached the unit circumference, (2.1) would obviously be false, and the proof could not be carried out. The counter-examples in the following section need such rapidly oscillating curves.

(3.24) COROLLARY. *For almost all values of* ξ, $0 \leqq \xi \leqq 2\pi$,

$$\lim_{P \to L_\alpha(\xi)e^{i\xi}} u(P) = U(\xi)$$

for almost all values of α, $\alpha = \alpha(\xi)$, $0 \leqq \alpha < \pi/2$.

PROOF. Define the function $X(\alpha, \xi)$ as

$$X(\alpha, \xi) = \begin{cases} 1 & \text{if } \lim_{P \to L_\alpha(\xi)e^{i\xi}} u(P) \text{ fails to exist,} \\ 0 & \text{if } \lim_{P \to L_\alpha(\xi)e^{i\xi}} u(P) \text{ does exist.} \end{cases}$$

It is not difficult to show that $X(\alpha, \xi)$ is a measurable function of the variables α and ξ. It follows from Theorem (3.5) that for fixed α, $X(\alpha, \xi) = 0$ for almost ξ. It follows immediately from Fubini's theorem that for almost any fixed value of ξ, $X(\alpha, \xi) = 0$ for almost all α. This proves the corollary.

4. **Counter-examples.** In view of the remarks made at the beginning of §3, a counter-example to Priwaloff's theorem will be any potential function (3.4) whose associated mass distribution function, $F(Q)$, satisfies (3.3) but which fails to have a non-tangential limit at a set of points on the unit circumference of positive measure.

The following counter-example is essentially the one described by Zygmund in a letter to Tamarkin in 1943.

(4.1) EXAMPLE. Consider the potential function

$$w_1(P) = \int\int g(P, Q)dF_1(Q). \tag{4.2}$$

Suppose the mass distribution function $F_1(Q)$ is totally discreet; that

is, the mass is concentrated at a denumerable set E of points converging to the unit circumference. For example, let

$$E = E_1 + E_2 + E_3 + \cdots + E_n + \cdots$$

where E_n consists of n points equally distributed on the circumference of the circle of radius $\rho_n = 1 - 1/n$ with one of the n points lying on the positive x-axis. At each point of E_n place a mass of $1/2^n$. Then the total mass contained in the set E_n is $n/2^n$ and we have

$$\iint (1 - \rho) dF_1(Q) = \sum_{n=1}^{\infty} \frac{1}{n} \frac{n}{2^n} = 1.$$

Thus (3.3) is satisfied for the mass distribution function $F_1(Q)$.

For ψ such that $\tan^{-1} 2\pi < \psi < \pi/2$, it is not difficult to show that for sufficiently large n at least one point from each of the sets E_n, E_{n+1}, E_{n+2}, $\cdots$ lies within the sector $S_\psi(\xi)$ for any value of ξ, $0 \leqq \xi < 2\pi$.

Choose a path approaching $e^{i\xi}$ which passes through points of the sets E_n, E_{n+1}, $\cdots$ and remains within the sector $S_\psi(\xi)$. At each of these points $w_1(P) = +\infty$. Thus we have that

$$\limsup_{P \to S_\psi(\xi) e^{i\xi}} w_1(P) = +\infty \tag{4.3}$$

for all values of ξ, $0 \leqq \xi < 2\pi$.

Since $w_1(P)$ satisfies (3.3) yet fails to have a finite non-tangential limit at every point of the unit circumference, we have a counter-example to Priwaloff's theorem.

It is interesting to note that

$$\liminf_{P \to S_\psi(\xi) e^{i\xi}} w_1(P) = 0 \tag{4.4}$$

for almost all values of ξ, $0 \leqq \xi < 2\pi$. For Littlewood showed that the radial limit of a potential of the form (4.2) is zero at almost all points of the unit circumference [1, Lemma 4].

If, instead of considering sectorial approaches to the unit circumference, we restrict the path of approach to lie between any two distinct curves tangent to each other at the point $e^{i\xi}$, the limit of the potential function (3.4) may still fail to exist at all points of the unit circumference. It is easy to construct such a function by simply placing a denumerable infinity of masses on each of the circles of radius ρ_n and at the same time preserve (3.3).

The final counter-example is that of a bounded superharmonic function whose associated mass distribution function $F(Q)$ is absolutely

continuous. That means, for any Borel measurable subset e of the unit circle

$$F(e) = \int\int_e f(Q)dQ \tag{4.5}$$

where $f(Q)$ is a non-negative Lebesgue integrable function called the density function associated with the given superharmonic function (see [3, 4.33]).

(4.6) EXAMPLE. Let the circles $C_{n_i}(P_{n_i}, R_n)$, $i=1, 2, \cdots, n$, be the n circles whose centers are at the points of the set E_n in (4.1) and whose radii are $R_n=1/2^n$. Consider the following function defined in the unit circle.

$$w_2(P) = \begin{cases} 0 \text{ for } P \text{ outside the circles } C_{n_i}; \\ \qquad i = 1, 2, \cdots, n; n = 1, 2, \cdots, \\ 1 - (R/R_n)^{1/2^n} \quad \text{for } P \text{ within one of the circles } C_{n_i} \\ \qquad \text{where } P \text{ has polar coordinates } (R, \theta) \\ \qquad \text{with respect to the point } P_{n_i}. \end{cases} \tag{4.7}$$

Outside the circles C_{n_i}, $w_2(P)$ is harmonic since $\Delta w_2(P)=0$ trivially. For P within any one of the circles C_{n_i} the Laplacian is

$$\begin{aligned} \Delta w_2(P) = \Delta w_2(R, \theta) &= \frac{\partial^2 w_2}{\partial R^2} - \frac{1}{R}\frac{\partial w_2}{\partial \theta} - \frac{\partial^2 w_2}{\partial \theta^2} \\ &= \frac{-1}{2^{2n}R_n^{1/2^n}}\left[R^{(1/2^n)-2}\right]. \end{aligned} \tag{4.8}$$

Since the Laplacian is non-positive in the unit circle, our function (4.7) is superharmonic in the unit circle (see [3, 4.1]).

The superharmonic function (4.7) takes on the value 1 at the points P_{n_i} of E and the value 0 in the space between the non-overlapping circles C_{n_i}. Hence if we choose a path approaching $e^{i\xi}$ through the points of E as in (4.1), we find that $w_2(P)$ will oscillate between the values 0 and 1. Thus at every point of the unit circumference (4.7) fails to have a non-tangential limit.

According to Rado [3, 6.2] the density function associated with a superharmonic function is given by the formula

$$f(Q) = (-1/2\pi)\,\Delta w_2(Q). \tag{4.9}$$

Thus for the superharmonic function (4.10) the density function is given by

$$(4.10)\qquad f(Q)=\begin{cases}0 \text{ for } Q \text{ outside the circles } C_{n_i},\\ \dfrac{1}{2\pi}\cdot\dfrac{1}{2^{2n}}\cdot\dfrac{1}{R_n^{1/2^n}}\cdot R^{(1/2^n)-2}\\ \qquad\text{where } Q \text{ is within } C_{n_i} \text{ and has polar coordinates}\\ \qquad(R,\theta) \text{ with respect to } P_{n_i}.\end{cases}$$

Since $f(Q)$ in (4.10) is summable the mass distribution function, $F(Q)$ is absolutely continuous.

In order to show that (4.7) is our desired counter-example, it remains to show that (3.3) is satisfied by $F(Q)$. But in view of (4.5) we may write (3.3) as

$$(4.11)\qquad \iint (1-\rho)f(Q)dQ < +\infty.$$

Since $f(Q)=0$ outside the circles C_{n_i}, $i=1, 2, \cdots, n$; $n=1, 2, \cdots$, we may write the integral (4.11) as

$$(4.12)\qquad \iint (1-\rho)f(Q)dQ=\sum_{n=1}^{\infty} n\iint_{C_{n_i}}(1-\rho)f(Q)dQ.$$

For Q in C_{n_1} we have that $1-\rho\leqq 1/n+1/2^n$; hence it follows from (4.10) and (4.12) that

$$\begin{aligned}&\iint (1-\rho)f(Q)dQ\\ &\qquad\leqq\sum_{n=1}^{\infty}\left\{1+\frac{n}{2^n}\right\}\left\{\int_0^{2\pi}\int_0^{R_n}\frac{R^{(1/2^n)-2}}{2\pi 2^n R_n^{1/2^n}}RdRd\theta\right\}\\ &\qquad\leqq\sum_{n=1}^{\infty}\left\{1+\frac{n}{2^n}\right\}\left\{\frac{1}{2^n}\right\}<+\infty.\end{aligned}$$

Thus (4.11) is satisfied and (4.7) is the desired counter-example.

Bibliography

1. J. E. Littlewood, *On functions subharmonic in a circle.* II, Proc. London Math. Soc. (2) vol. 28 (1928) pp. 383–394.

2. I. Priwaloff, *Sur un problème limite des fonctions sous-harmoniques.* Rec. Math. (Mat. Sbornik) N.S. vol. 41 (1934) pp. 3–10.

3. T. Rado, *Subharmonic functions,* Ergebnisse der Mathematik und ihrer Grenzgebiete, Berlin, Springer, 1937.

4. G. C. Evans, *The logarithmic potential,* Amer. Math. Soc. Colloquium Publications, vol. 6, 1927.

Pomona College

GENERALIZATION OF A THEOREM OF OSGOOD TO THE CASE OF CONTINUOUS APPROXIMATION[1]

A. M. OSTROWSKI

In the historical development of the theory of convergence the following theorem of Osgood[2] has played a very important role: *Suppose that the functions $f_\nu(x)$ are continuous in the interval (a, b) and with $\nu \to \infty$ converge to a limit $f(x)$ which is also continuous in (a, b); then to any $\epsilon > 0$ there exists a subinterval (a', b') of (a, b) and an integer n_0 such that*

$$|f_\nu(x) - f(x)| < \epsilon$$

is valid in (a', b') for all $\nu > n_0$.

Although since 1897 a great number of results in this direction have been found, Osgood's theorem has not been as yet completely superseded and provides sometimes a useful means in dealing with convergent sequences. In what follows I prove an analogous theorem for the case of *continuous* convergence.

THEOREM. *Let $f(t, x)$ be a continuous function of the point (t, x) for $a < x < b$ and $t \geqq T$ and suppose that we have*

$$\lim_{t \to \infty} f(t, x) = f(x)$$

where $f(x)$ is also continuous in (a, b); then for any $\epsilon > 0$ there exists a subinterval J of (a, b) and a number T_0 such that we have for $x < J$, $t \geqq T_0$:

$$|f(t, x) - f(x)| < \epsilon \qquad (t \geqq T_0,\ x < J). \tag{1}$$

The proof of our theorem follows easily from the following

LEMMA. *If $f(t, x)$ is a continuous function of the point (t, x) for $t \geqq T$ and $a'' < x < b''$, where T is a positive integer, then a sequence of positive numbers t_ν tending to ∞ can be found such that to each $t \geqq T$ there corresponds a t_ν tending to ∞ with t, for which we have*

$$|f(t, x) - f(t_\nu, x)| \leqq 1/t.$$

PROOF OF THE LEMMA. Since $f(t, x)$ is *uniformly* continuous in any

Received by the editors July 1, 1949.

[1] The preparation of this paper was sponsored by the Office of Naval Research.

[2] W. F. Osgood, *Non-uniform convergence and the integration of series term by term*, Amer. J. Math. vol. 19 (1897) p. 161, the "Fundamental Theorem."

rectangle $a'' \leqq x \leqq b''$, $n \leqq t \leqq n+1$, for $n \geqq T$, there corresponds to any integer $n \geqq T$ a positive integer N_n such that we have

$$|f(t, x) - f(t_0, x)| \leqq \frac{1}{n+1} \tag{2}$$

if t and t_0 lie between n and $n+1$ and $|t-t_0| \leqq 1/N_n$, $a'' \leqq x \leqq b''$. We subdivide all intervals $\langle n, n+1\rangle$ with $n \geqq T$ in N_n equal parts and denote the division points from $t_1 = T$ on by $t_1, t_2, \cdots$. Then for each $t \geqq T$ there exists a t_ν from the interval between $[t]$ and $[t]+1$ such that $t_\nu \leqq t < t_{\nu+1}$ and therefore by (2) for all x in $\langle a'', b''\rangle$

$$|f(t, x) - f(t_\nu, x)| \leqq \frac{1}{[t_\nu]+1} \leqq \frac{1}{t}.$$

Our lemma is proved.

PROOF OF THE THEOREM. Without loss of generality T in the theorem can be supposed a positive integer. Apply the lemma to a closed interval $\langle a'', b''\rangle$ contained in (a, b) and form for this closed interval and the function $f(t, x)$ the sequence t_ν of the lemma. In applying Osgood's theorem to the sequence of functions $f(t_\nu, x)$, to a given $\epsilon > 0$ we find a subinterval J of $\langle a'', b''\rangle$ and an integer n_0 such that

$$|f(t_\nu, x) - f(x)| \leqq \epsilon/2 \qquad (\nu \geqq n_0,\ x < J). \tag{3}$$

Take then T_0 such that $1/T_0 < \epsilon/2$ and such that for each $t \geqq T_0$, a t_ν corresponding to t by the lemma has an index $\geqq n_0$. Then we have

$$|f(t, x) - f(t_\nu, x)| \leqq 1/t \leqq 1/T_0 < \epsilon/2 \tag{4}$$

for all x from $\langle a'', b''\rangle$ and the assertion (1) follows from the inequalities (3) and (4). Our theorem is proved.

Both the theorem and the lemma proved remain valid if t goes to ∞ through a set M which has a closed intersection with any finite interval. The necessary modifications in the proof are obvious. The case of a finite limiting point for t is reduced to the case treated above by a linear transformation.

NATIONAL BUREAU OF STANDARDS AND
UNIVERSITY OF BASLE

UNIFORM APPROXIMATIONS TO A CLASS OF BESSEL FUNCTIONS

ALFRED M. PEISER

1. Consider a function $t(x, y)$, defined in $x \geqq 0$, $y \geqq 0$, with continuous first partial derivatives, which satisfies the differential equation

$$\frac{\partial^2 t}{\partial x \partial y} + \frac{\partial t}{\partial x} + \frac{\partial t}{\partial y} = 0. \tag{1}$$

It is well known[1] that $t(x, y)$ is uniquely determined when boundary values $t(x, 0)$ and $t(0, y)$ are specified. By a straight-forward application of Laplace transforms, one may obtain (formally) the solution

$$\begin{aligned} t(x, y) &= t(x, 0)e^{-y} + \int_0^x t(u, 0)K_1(x - u, y)du \\ &\quad + \int_0^y \left[t(0, v) + \frac{\partial t(0, v)}{\partial v} \right] K_0(y - v, x)dv, \end{aligned} \tag{2}$$

where

$$K_n(x, y) = (y/x)^{n/2}e^{-x-y}I_n(2(xy)^{1/2}), \qquad n = 0, 1, 2, \cdots, \tag{3}$$

and I_n is the Bessel function of order n and imaginary argument. It is not difficult to verify, by direct differentiation, that (2) satisfies (1).

In certain applications,[2] $t(0, y)$ is given explicitly but $t(x, 0)$ is given in terms of the values of $t(x, y)$ at some other (fixed) value of y, namely,

$$t(x, 0) = a + bt(x, P) \text{ or } t(x, 0) = a + bt(S - x, P),$$

where a, b, P, and S are constants. In such cases, one readily obtains, from (2), an integral equation for $t(x, 0)$ of the form

$$\left.\begin{matrix} t(x, 0) \\ t(S - x, 0) \end{matrix}\right\} = F(x) + bt(x, 0)e^{-P} + b\int_0^x t(u, 0)K_1(x - u, P)du, \tag{4}$$

Presented to the Society, April 30, 1949; received by the editors April 9, 1949 and, in revised form, August 8, 1949.

[1] J. D. Tamarkin, *Partial differential equations*, Brown University Notes, 1941, p. 21.

[2] See for example, H. Hausen, *Naherungsverfahren zur Berechnung des Warmeaustausches in Regeneratoren*, Zeitschrift für Angewandte Mathematik und Mechanik vol. 11 (1931) pp. 105–114.

where $F(x)$ is known.

In the evaluation of (2) for particular boundary conditions,[3] or in the solution of integral equations of the type (4), it is sometimes convenient to replace $K_0(x, y)$ and $K_1(x, y)$ by approximating functions which are more suitable for numerical or analytical work. For large values of the arguments, one could, of course, use the asymptotic formulas for the Bessel functions,[4] while for small values of the arguments, the power series expansions would suffice. It is desirable, however, particularly in the case of the integral equation (4), to obtain approximations to $K_0(x, y)$ and $K_1(x, y)$ which hold *uniformly* in $0 \leqq x < \infty$.

We shall show that, for suitably large values of y, K_0 and K_1 can be approximated uniformly in x by means of the normal probability function,

$$\Phi(x) = \frac{1}{(2\pi)^{1/2}} e^{-x^2/2}. \tag{5}$$

We find that

$$\lim_{y\to\infty} (2y+1)^{1/2} K_0(y + 1 + x(2y+1)^{1/2}, y) = \Phi(x) \tag{6a}$$

and

$$\lim_{y\to\infty} (2y)^{1/2} K_1(y + x(2y)^{1/2}, y) = \Phi(x), \tag{6b}$$

and obtain, for $x \geqq 0$, the inequalities

$$\left| K_0(x, y) - \frac{1}{(2y+1)^{1/2}} \Phi\left(\frac{x-y-1}{(2y+1)^{1/2}}\right)\right| < \frac{0.321}{2y+1} + \frac{5.101}{(2y+1)^{3/2}} + \frac{0.894}{(2y+1)^{1/2}} e^{-y/5}, \qquad y \geqq 10, \tag{7a}$$

$$\left| K_1(x, y) - \frac{1}{(2y)^{1/2}} \Phi\left(\frac{x-y}{(2y)^{1/2}}\right)\right| < \frac{0.167}{y} + \frac{0.800}{y^{3/2}} + \frac{0.694}{y^{1/2}} e^{-y/5}, \qquad y \geqq 10. \tag{7b}$$

Examination of tabulated values of the functions indicates that

[3] See, for example, P. Bateman, *Partial differential equations*, Dover, 1944, p. 125; A. M. Peiser, *The unsteady state cooling of beds*, submitted for publication to Industrial and Engineering Chemistry.

[4] G. N. Watson, *Bessel functions*, Cambridge, 1945, p. 203.

these error estimates exceed the actual errors by at most about a factor of two. Better approximations to K_0 and K_1 may be obtained by means of asymptotic series involving the successive derivatives of $\Phi(x)$. Series of this type are given in §6.

2. The Laplace transforms. We define $K_n(x, y)$ to be zero when $x<0$ and consider the bilateral Laplace transforms

$$k_n(\lambda, y) = \int_{-\infty}^{\infty} e^{-\lambda x} K_n(x, y)\,dx, \qquad \Re\lambda > -1,$$

$$\phi(\lambda) = \int_{-\infty}^{\infty} e^{-\lambda x}\Phi(x)\,dx = e^{\lambda^2/2}. \tag{8}$$

By standard integration formulas,[5] we find

$$k_0(\lambda, y) = \exp\left\{-y + \frac{y}{\lambda+1}\right\}\Big/(\lambda+1), \qquad \Re\lambda > -1,$$

$$k_1(\lambda, y) = \exp\left\{-y + \frac{y}{\lambda+1}\right\} - \exp\{-y\}, \quad \Re\lambda > -1.$$

Let

$$H_0(x, y) = (2y+1)^{1/2}K_0(y+1+x(2y+1)^{1/2}, y), \qquad x \geqq -\frac{y+1}{(2y+1)^{1/2}}, \tag{9a}$$

$$H_1(x, y) = (2y)^{1/2}K_1(y+x(2y)^{1/2}, y), \qquad x \geqq -(y/2)^{1/2}. \tag{9b}$$

Then, if $h_n(\lambda, y)$ denotes the bilateral transform of $H_n(x, y)$, we have[5]

$$\begin{aligned} h_0(\lambda, y) &= k_0(\lambda/(2y+1)^{1/2}, y)\exp\left\{\frac{\lambda(y+1)}{(2y+1)^{1/2}}\right\} \\ &= \exp\left\{-y + \frac{y}{1+\lambda/(2y+1)^{1/2}} + \frac{\lambda(y+1)}{(2y+1)^{1/2}}\right\}\Big/(1+\lambda/(2y+1)^{1/2}), \end{aligned} \tag{10a}$$

$$\begin{aligned} h_1(\lambda, y) &= k_1(\lambda/(2y)^{1/2}, y)\exp\{\lambda(y/2)^{1/2}\} \\ &= \exp\{-y+\lambda(y/2)^{1/2}\}\cdot\left[\exp\left\{\frac{y}{1+\lambda/(2y)^{1/2}}\right\} - 1\right]. \end{aligned} \tag{10b}$$

[5] R. V. Churchill, *Modern operational mathematics in engineering*, New York, 1944, pp. 294, 299, formula 81.

It follows from (8) that

$$\lim_{y\to\infty} h_n(\lambda, y) = \phi(\lambda), \qquad n = 0, 1,$$

uniformly in each bounded region of the complex λ-plane. From this result and from standard theorems for the Laplace transform, one can deduce (6a) and (6b). It is not necessary to give details at this point since we are going to prove the stronger results (7a) and (7b).

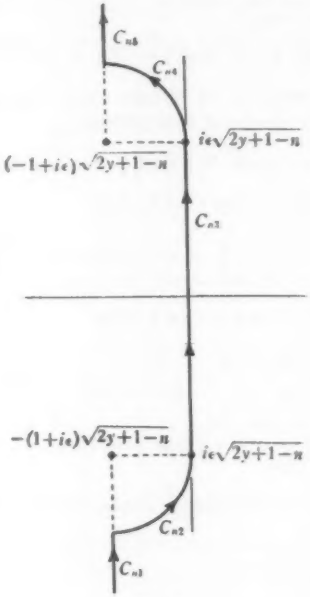


FIG. 1

By means of the complex inversion formula for the Laplace transform,[6] we can write

$$H_n(x, y) = \frac{1}{2\pi i}\int_{C_n} e^{\lambda x} h_n(\lambda, y)\,d\lambda, \qquad x \geqq -\frac{y+1-n}{(2y+1-n)^{1/2}}, \quad n = 0, 1,$$

where C_n is any vertical line $\Re\lambda = \alpha$, with $\alpha > -(2y+1-n)^{1/2}$, and where the principal value of the integral is to be taken. Now, $h_n(\lambda, y)$

[6] D. V. Widder, *The Laplace transform*, Princeton, 1946, p. 241.

is analytic at each point of the finite λ-plane except for an essential singularity at $\lambda = -(2y+1-n)^{1/2}$. By a simple application of Cauchy's theorem, we can replace the contour C_n by the contour shown in Fig. 1. We assume $0<\epsilon<1$. If we let

$$I_{nj} = \frac{1}{2\pi i}\int_{\sigma_{nj}} e^{\lambda x} h_n(\lambda, y)\,d\lambda, \tag{11}$$

we can write, for $n=0$, 1, and $x \geqq -(y+1-n)/(2y+1-n)^{1/2}$,

$$\begin{aligned} &|H_n(x, y) - \Phi(x)| \\ &\quad \leqq |I_{n3} - \Phi(x)| + |I_{n1} + I_{n5}| + |I_{n2} + I_{n4}|. \end{aligned} \tag{12}$$

In the following sections we obtain upper bounds for each of the quantities on the right-hand side of (12).

3. **Preliminary lemmas.** We first prove the following lemmas:

LEMMA 1. *If* $a>0$, $-\epsilon<u<\epsilon<1$, *then*

$$\exp\left\{-\frac{a}{1+iu}\right\} = e^{-a} + iaue^{-a} + \frac{\theta u^2}{1-\epsilon}e^{-a/2}, \quad |\theta| < 1.$$

LEMMA 2. *If* $b>0$, $-\epsilon<u<\epsilon<1$, *then*

$$f(iu) = \frac{1}{1+iu}\exp\left\{\frac{1+iu}{2} - \frac{b+1}{2(1+iu)}\right\} = e^{-b/2} + \frac{ibu}{2}e^{-b/2}$$
$$+ \frac{\theta u^2 e^{-b/4}}{\pi(1-\epsilon)}\left[\frac{e^{3/4}}{(2\epsilon)^{1/2}}\arccos(\epsilon-1) + \frac{8e^{\epsilon/2-1/4}}{b(2-\epsilon)}\right], \quad |\theta| < 1.$$

To prove Lemma 1, we apply Taylor's theorem in the form

$$\exp\left\{-\frac{a}{1+iu}\right\} = e^{-a} + iaue^{-a} - \frac{u^2}{2\pi i}\int_C \frac{\exp\{-a/(1+w)\}}{w^2(w-iu)}\,dw, \tag{13}$$

where C is the circle $|w| = \rho$, $\epsilon<\rho<1$. For w on C, $w=\rho e^{i\phi}$, we have

$$\left|\frac{\exp\{-a/(1+w)\}}{w^2(w-iu)}\right| \leqq \frac{\exp\{-a\Re(1/(1+w))\}}{\rho^2(\rho-\epsilon)}, \quad \Re = \text{real part.}$$

But, for $\rho<1$,

$$\Re\frac{1}{1+w} = \frac{1+\rho\cos\phi}{1+\rho^2+2\rho\cos\phi} \geqq \frac{1}{1+\rho},$$

so that

$$\left|\frac{\exp\{-a/(1+w)\}}{w^2(w-iu)}\right| \leqq \frac{\exp\{-a/(1+\rho)\}}{\rho^2(\rho-\epsilon)}.$$

Lemma 1 now follows easily from (13) if we let $\rho \to 1$.

To prove Lemma 2, we apply Taylor's theorem in the form

$$f(iu) = e^{-b/2} + \frac{ibu}{2}e^{-b/2} - \frac{u^2}{2\pi i}\int_\Gamma \frac{f(w)}{w^2(w-iu)}dw, \tag{14}$$

where Γ is the contour shown in Fig. 2. We note that for w on Γ,

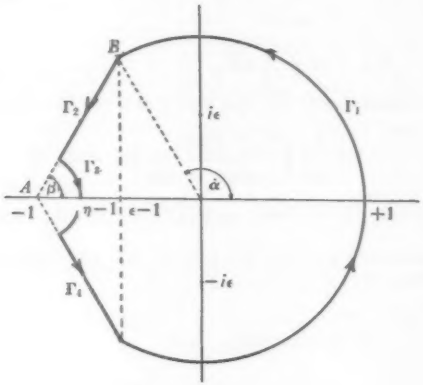


FIG. 2

$|w-iu| > 1-\epsilon$, and that $AB = (2\epsilon)^{1/2}$, $\cos\beta = (\epsilon/2)^{1/2}$, $\cos\alpha = \epsilon - 1$.

For w on Γ_1, $w = e^{i\phi}$, $\cos\phi > \epsilon - 1$, and we have

$$|f(w)| = (2+2\cos\phi)^{-1/2}\exp\left\{\frac{1+\cos\phi}{2} - \frac{b+1}{2}\Re\frac{1}{1+e^{i\phi}}\right\}$$

$$\leqq \frac{1}{(2\epsilon)^{1/2}}\exp\left\{1 - \frac{b+1}{4}\right\}.$$

Hence

$$\left|\frac{1}{2\pi i}\int_{\Gamma_1}\frac{f(w)dw}{w^2(w-iu)}\right| \leqq \frac{e^{3/4-b/4}}{\pi(2\epsilon)^{1/2}(1-\epsilon)}\arccos(\epsilon-1). \tag{15}$$

For w on Γ_2, $w = -1 + re^{i\theta}$, $|w|^2 > 1-\epsilon/2$, $\eta < r < (2\epsilon)^{1/2}$, and

$$|f(w)| = \frac{1}{r}\exp\left\{\frac{r}{2}\cos\beta - \frac{b+1}{2r}\cos\beta\right\}$$
$$< \frac{1}{r}\exp\left\{\frac{\epsilon}{2} - \frac{b+1}{2r}\left(\frac{\epsilon}{2}\right)^{1/2}\right\}.$$

Thus

$$\left|\frac{1}{2\pi i}\int_{\Gamma_3}\frac{f(w)dw}{w^2(w-iu)}\right|$$
$$(16)\qquad \leqq \frac{e^{\epsilon/2}}{\pi(1-\epsilon)(2-\epsilon)}\int_{\eta}^{(2\epsilon)^{1/2}}\frac{1}{r}\exp\left\{-\frac{b+1}{2r}\left(\frac{\epsilon}{2}\right)^{1/2}\right\}dr$$
$$< \frac{4e^{\epsilon/2-(b+1)/4}}{\pi(1-\epsilon)(2-\epsilon)b}.$$

The same estimate holds for the integral over Γ_4. It is easy to show

$$(17)\qquad \left|\frac{1}{2\pi i}\int_{\Gamma_3}\frac{f(w)}{w^2(w-iu)}dw\right| \to 0 \qquad \text{as } \eta \to 0.$$

Combining (14), (15), (16), and (17), we obtain Lemma 2.

4. **Error estimate for K_1.** To obtain the estimate (7b), we note first, from (10b), (11), and Fig. 1, that

$$I_{13} = \frac{1}{2\pi}\int_{-\epsilon(2y)^{1/2}}^{\epsilon(2y)^{1/2}} e^{ivx}h_1(iv, y)dv$$
$$= \frac{1}{2\pi}\int_{-\epsilon(2y)^{1/2}}^{\epsilon(2y)^{1/2}} e^{ivx}\left[\exp\left\{-\frac{v^2}{2}\cdot\frac{1}{1+iv/(2y)^{1/2}}\right\}\right.$$
$$\left. - \exp\left\{-y + iv(y/2)^{1/2}\right\}\right]dv.$$

We apply Lemma 1 with $a = v^2/2$, $u = v/(2y)^{1/2}$, to obtain

$$I_{13} = \frac{1}{2\pi}\int_{-\epsilon(2y)^{1/2}}^{\epsilon(2y)^{1/2}} e^{ivx}\left[e^{-v^2/2} + \frac{iv^3}{2(2y)^{1/2}}e^{-v^2/2}\right.$$
$$\left. + \frac{\theta v^3}{2y(1-\epsilon)}e^{-v^2/4} - e^{-y+iv(y/2)^{1/2}}\right]dv.$$

Combining (5) and (8), we have

$$(18)\qquad \Phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ivx-v^2/2}dv,$$

and we can write

$$|I_{13}-\Phi(x)| \leqq \frac{1}{2\pi}\int_{|v|>\epsilon(2y)^{1/2}} e^{-v^2/2}dv + \frac{1}{4\pi(2y)^{1/2}}\int_{-\infty}^{\infty} |v|^3 e^{-v^2/2}dv + \frac{1}{4\pi y(1-\epsilon)}\int_{-\infty}^{\infty} v^2 e^{-v^2/4}dv + \frac{\epsilon(2y)^{1/2}}{\pi} e^{-y}.$$

With the inequality

$$\int_{|v|>\delta} e^{-v^2/2}dv < \frac{1}{\delta}\int_{|v|>\delta} ve^{-v^2/2}dv = \frac{2}{\delta} e^{-\delta^2/2},$$

we find

$$(19)\quad |I_{13}-\Phi(x)| \leqq \frac{1}{\pi(2y)^{1/2}} + \frac{1}{\pi^{1/2}(1-\epsilon)y} + \frac{e^{-\epsilon^2 y}}{\pi\epsilon(2y)^{1/2}} + \frac{\epsilon(2y)^{1/2}}{\pi} e^{-y}.$$

For the integrals I_{11} and I_{15} we make the change of variables $\lambda=(2y)^{1/2}(-1+iv)$. From (10b) and (11) we obtain

$$I_{11}+I_{15} = \frac{(2y)^{1/2}}{2\pi} e^{-2y-x(2y)^{1/2}}\int_{|v|>1+\epsilon} e^{iv(y+x(2y)^{1/2})}\left[e^{-iy/v}-1\right]dv.$$

For real u, we can write $e^{iu}=1+iu+\theta u^2/2$, $|\theta|<1$. Hence

$$I_{11}+I_{15} = \frac{(2y)^{1/2}}{2\pi} e^{-2y-x(2y)^{1/2}}\int_{|v|>1+\epsilon} e^{iv(y+x(2y)^{1/2})}\left[-\frac{iy}{v}+\frac{\theta y^2}{2v^2}\right]dv.$$

Since $x \geqq -(y/2)^{1/2}$, we have

$$|I_{11}+I_{15}| \leqq \frac{(2y)^{1/2}}{2\pi} e^{-y}\left[\left|\int_{|v|>1+\epsilon} ye^{iv(y+x(2y)^{1/2})}\frac{dv}{v}\right| + \frac{y^2}{2}\int_{|v|>1+\epsilon}\frac{dv}{v^2}\right].$$

But for $\gamma>0$,

$$\left|\int_{|v|>\gamma} e^{iv}\frac{dv}{v}\right| = \left|\int_{|v|>\gamma}\frac{\sin v}{v}dv\right| < \pi,$$

so that

$$(20)\qquad |I_{11}+I_{15}| \leqq \frac{(2y)^{1/2}}{2\pi} e^{-y}\left\{\pi y + \frac{y^2}{1+\epsilon}\right\}.$$

To estimate I_{14}, let $\lambda=(2y)^{1/2}(-1+i\epsilon+e^{i\psi})$. Then

$$I_{14} = \frac{(2y)^{1/2}}{2\pi} e^{-y} \int_0^{\pi/2} \exp\{(y + x(2y)^{1/2})(-1 + i\epsilon + e^{i\phi})\} \cdot [e^{y/(i\epsilon+e^{i\phi})} - 1] e^{i\phi} d\phi.$$

Since $x \geqq -(y/2)^{1/2}$,

$$|I_{14}| \leqq \frac{(2y)^{1/2}}{2\pi} e^{-y} \int_0^{\pi/2} \exp\{(y + x(2y)^{1/2})(\cos\phi - 1)\} \cdot \left[1 + \exp\left\{\frac{y\cos\phi}{1 + \epsilon^2 + 2\epsilon\sin\phi}\right\}\right] d\phi$$

$$\leqq \frac{(2y)^{1/2}}{2\pi} e^{-y} \int_0^{\pi/2} \left[1 + \exp\left\{\frac{y\cos\phi}{1+\epsilon^2}\right\}\right] d\phi.$$

With the change of variable $\sin(\phi/2) = u$, we find

$$\begin{aligned} &\int_0^{\pi/2} \exp\left\{\frac{y\cos\phi}{1+\epsilon^2}\right\} d\phi \\ &\quad = 2\exp\left\{\frac{y}{1+\epsilon^2}\right\} \int_0^{1/2^{1/2}} \frac{\exp\{-2yu^2/(1+\epsilon^2)\}\, du}{(1-u^2)^{1/2}} \\ &\quad < 2^{3/2} \exp\left\{\frac{y}{1+\epsilon^2}\right\} \int_0^{\infty} \exp\left\{\frac{-2yu^2}{1+\epsilon^2}\right\} du \\ &\quad = \left(\frac{\pi(1+\epsilon^2)}{y}\right)^{1/2} \exp\left\{\frac{y}{1+\epsilon^2}\right\}. \end{aligned} \tag{21}$$

Thus,

$$|I_{14}| \leqq \frac{(2y)^{1/2}}{4} e^{-y} + \left(\frac{1+\epsilon^2}{2\pi}\right)^{1/2} \exp\left\{-\frac{\epsilon^2 y}{1+\epsilon^2}\right\}. \tag{22}$$

The same estimate may be obtained for I_{12}.

To obtain (7b), we take $y \geqq 10$ and $\epsilon = 1/2$. Then (19), (20), and (22) become, respectively,

$$\begin{aligned} |I_{13} - \Phi(x)| &\leqq \frac{1}{\pi(2y)^{1/2}} + \frac{2}{\pi^{1/2}y} + \frac{2^{1/2}}{\pi y^{1/2}} e^{-y/4} + \frac{y^{1/2}}{\pi 2^{1/2}} e^{-y} \\ &< \frac{1}{\pi(2y)^{1/2}}(1 + ye^{-y}) + \frac{2}{\pi^{1/2}y} + \frac{1}{\pi}\left(\frac{2}{ye}\right)^{1/2} e^{-y/5} \\ &< \frac{0.2252}{y^{1/2}} + \frac{1.130}{y} + 0.0864e^{-y/5}; \end{aligned} \tag{23}$$

$$(24)\qquad |I_{11}+I_{15}| \leqq \frac{1}{\pi(2y)^{1/2}}e^{-v}\left(\pi y^2+\frac{2}{3}y^3\right) < \frac{0.0101}{y^{1/2}};$$

$$(25)\qquad \begin{aligned}|I_{12}|+|I_{14}| &\leqq \frac{1}{(2y)^{1/2}}\cdot ye^{-v}+\left(\frac{5}{2\pi}\right)^{1/2}e^{-v/5}\\ &< \frac{0.0004}{y^{1/2}}+0.893e^{-v/5}.\end{aligned}$$

Combining (12), (23), (24), and (25), we obtain finally

$$|H_1(x,y)-\Phi(x)| < \frac{0.236}{y^{1/2}}+\frac{1.130}{y}+0.980e^{-v/5}.$$

Equation (7b) now follows from (9b) and this inequality.

5. **Error estimate for K_0.** The procedure used to obtain (7b) may be followed with obvious modifications to obtain (7a). Accordingly, we shall only sketch the proof of (7a). From (10a), (11), and Fig. 1, we have

$$\begin{aligned}I_{03} &= \frac{1}{2\pi}\int_{-\epsilon(2y+1)^{1/2}}^{\epsilon(2y+1)^{1/2}} e^{ivx}h_0(iv,y)\,dv\\ &= \frac{1}{2\pi}\int_{-\epsilon(2y+1)^{1/2}}^{\epsilon(2y+1)^{1/2}} e^{ivx}\exp\left\{-\frac{v^2+1}{2(1+iv/(2y+1)^{1/2})}\right.\\ &\qquad\left.+\frac{1+iv/(2y+1)^{1/2}}{2}\right\}\frac{dv}{1+iv/(2y+1)^{1/2}}.\end{aligned}$$

We apply Lemma 2 with $b=v^3$, $u=v/(2y+1)^{1/2}$, to obtain

$$\begin{aligned}I_{03} = \frac{1}{2\pi}\int_{-\epsilon(2y+1)^{1/2}}^{\epsilon(2y+1)^{1/2}} e^{ivx}\Bigg[&e^{-v^2/2}+\frac{iv^3}{2(2y+1)^{1/2}}e^{-v^2/2}\\ &+\frac{\theta e^{-v^2/4}}{\pi(2y+1)(1-\epsilon)}\left(\frac{e^{3/4}}{(2\epsilon)^{1/2}}\arccos(\epsilon-1)\cdot v^2+\frac{8e^{\epsilon/2-1/4}}{2-\epsilon}\right)\Bigg]dv.\end{aligned}$$

Proceeding as in §4, we find

$$(26)\qquad \begin{aligned}|I_{03}-\Phi(x)| &< \frac{1}{\pi(2y+1)^{1/2}}+\frac{\exp\{-(\epsilon^2/2)(2y+1)\}}{\pi\epsilon(2y+1)^{1/2}}\\ &\quad+\frac{2}{\pi^{3/2}(2y+1)(1-\epsilon)}\left[\frac{e^{3/4}}{(2\epsilon)^{1/2}}\arccos(\epsilon-1)+\frac{4e^{\epsilon/2-1/4}}{(2-\epsilon)}\right].\end{aligned}$$

For the integrals I_{01} and I_{05}, we make the change of variables

$\lambda=(2y+1)^{1/2}(-1+iv)$ and obtain

$$I_{01}+I_{05}=\frac{(2y+1)^{1/2}}{2\pi i}\exp\{-y-(y+1+x(2y+1)^{1/2})\}\cdot\int_{|v|>1+\epsilon}e^{iv(y+1+x(2y+1)^{1/2})}\left(1+\frac{\theta y}{v}\right)\frac{dv}{v},$$

with $|\theta|<1$. Since $x\geqq-(y+1)/(2y+1)^{1/2}$, we find, as in §4,

$$|I_{01}+I_{05}|<\frac{e^{-y}(2y+1)^{1/2}}{2\pi}\left(\pi+\frac{2y}{1+\epsilon}\right). \tag{27}$$

For I_{04} (and, similarly for I_{02}), we let $\lambda=(2y+1)^{1/2}(-1+i\epsilon+e^{i\phi})$, so that

$$I_{04}=\frac{(2y+1)^{1/2}}{2\pi}e^{-y}\int_0^{\pi/2}\exp\left\{(y+1+x(2y+1)^{1/2})(-1+i\epsilon+e^{i\phi})+\frac{y}{i\epsilon+e^{i\phi}}+i\phi\right\}\frac{d\phi}{i\epsilon+e^{i\phi}},$$

and

$$\begin{aligned}|I_{04}|&\leqq\frac{(2y+1)^{1/2}}{2\pi}e^{-y}\int_0^{\pi/2}\exp\left\{(y+1+x(2y+1)^{1/2})(\cos\phi-1)+\frac{y\cos\phi}{1+\epsilon^2+2\epsilon\sin\phi}\right\}\frac{d\phi}{(1+\epsilon^2+2\epsilon\sin\phi)^{1/2}}\\&<\frac{(2y+1)^{1/2}}{2\pi(1+\epsilon^2)^{1/2}}e^{-y}\int_0^{\pi/2}\exp\left\{\frac{y\cos\phi}{1+\epsilon^2}\right\}d\phi.\end{aligned}$$

Using (21), we obtain

$$|I_{04}|<\frac{(2y+1)^{1/2}}{(2\pi)^{1/2}(2y)^{1/2}}\exp\left\{-\frac{\epsilon^2y}{1+\epsilon^2}\right\}. \tag{28}$$

If we take $y\geqq10$ and $\epsilon=1/2$, the inequalities (26), (27), and (28) become, respectively,

$$\begin{aligned}|I_{03}-\Phi(x)|&<\frac{1}{\pi(2y+1)^{1/2}}+\frac{4}{\pi^{3/2}(2y+1)}\left[\frac{2\pi e^{3/4}}{3}+\frac{8}{3}\right]+\frac{2e^{-1/8-y/4}}{\pi(2y+1)^{1/2}}\\&<\frac{0.3184}{(2y+1)^{1/2}}+\frac{5.101}{2y+1}+0.0735e^{-y/5};\end{aligned} \tag{29}$$

$$(30) \qquad |I_{01} + I_{05}| < \frac{(2y+1)e^{-y}(\pi + 4y/3)}{2\pi(2y+1)^{1/2}} < \frac{0.0025}{(2y+1)^{1/2}};$$

$$(31) \qquad |I_{02}| + |I_{14}| < 0.820e^{-y/5}.$$

Combining (12), (29), (30), and (31), we obtain

$$|H_0(x, y) - \Phi(x)| < \frac{0.321}{(2y+1)^{1/2}} + \frac{5.101}{2y+1} + 0.894e^{-y/5}.$$

Equation (7a) now follows from (9a) and the last inequality.

6. **Series expansion for $K_n(x, y)$.** Each of the functions $K_n(x, y)$ can be expanded in terms of the successive derivatives of the normal probability function. For K_0 and K_1 ,we find, for example,

$$K_0(y + 1 + x(2y+1)^{1/2}, y)$$
$$= \frac{1}{(2y+1)^{1/2}}\Phi(x) - \frac{3y+1}{3(2y+1)^2}\Phi^{(3)}(x)$$
$$+ \left[\frac{4y+1}{4(2y+1)^{5/2}}\Phi^{(4)}(x) + \frac{(3y+1)^2}{18(2y+1)^{7/2}}\Phi^{(6)}(x)\right] + O\left(\frac{1}{y^2}\right);$$

$$K_1(y + x(2y)^{1/2}, y)$$
$$= \frac{1}{(2y)^{1/2}}\Phi(x) - \frac{1}{4y}\Phi^{(3)}(x)$$
$$+ \frac{1}{4y(2y)^{1/2}}\left[\Phi^{(4)}(x) + \frac{1}{4}\Phi^{(6)}(x)\right] + O\left(\frac{1}{y^2}\right).$$

Asymptotic series of this type are common in statistics.[7] They are related to the moments, $\int_0^\infty x^m K_n(x, y)dx$, of the function $K_n(x, y)$, and may be derived, for example, directly from the Laplace transform. Expansions of each of the functions $K_n(x, y)$, $n \geqq 2$, may be obtained in this manner.

HYDROCARBON RESEARCH, INC., NEW YORK, N. Y.

[7] H. Cramér, *Mathematical methods of statistics*, Princeton, 1946, pp. 227–229.

BANACH-HAUSDORFF LIMITS

W. F. EBERLEIN

1. **Introduction.** Let m denote the Banach lattice of bounded real sequences $x=(x_0, x_1, x_2, \cdots)$ with $\|x\|=\sup_n|x_n|$ and $x\geqq y$ if $x_n\geqq y_n$ for all n. Let S denote the shift operator $S(x_0, x_1, x_2, \cdots)=(x_1, x_2, \cdots)$. Banach[1] has established the existence of real-valued functionals $L(x)$ defined over m with the properties

(I) $L(ax+by)=aL(x)+bL(y)$ (a, b real);

(II) $L(1)=1$;

(III) $L(x)\geqq 0$ if $x\geqq 0$;

(IV) $L(Sx)=L(x)$.

The conditions (I)–(IV) imply that

$$(1)\qquad L_*(x)=\liminf_n x_n\leqq L(x)\leqq\limsup_n x_n=L^*(x),$$

whence a *Banach limit* of a convergent sequence is the ordinary limit.

Designating the class of regular Hausdorff[2] transformations by $\mathfrak{H}$, we term $L(x)$ a *Banach-Hausdorff* (*B-H*) *functional* or *limit* if in addition to (I)–(IV) it satisfies

(V) $L(Hx)=L(x)$ ($H\in\mathfrak{H}$).

The corresponding regularity property of B-H limits then takes the form: If the bounded sequence x is summable to a by some regular Hausdorff method, then $L(x)=a$ for every B-H functional L.

The obvious problem is the existence of B-H limits. Our solution yields simultaneously the existence and the *domain of uniqueness*—that is, the set of x in m on which all B-H functionals coincide.[3]

2. **Hausdorff lore.** We recall the following properties of the class $\mathfrak{H}$ of regular Hausdorff transformations: $\mathfrak{H}$ is the convex Abelian semi-group of linear transformations of m into itself defined by the Toeplitz matrices (a_{mn}), where

$$(2)\qquad a_{mn}=C_{m,n}\int_0^1 u^n(1-u)^{m-n}d\alpha(u)\qquad(n\leqq m)$$

$$=0\qquad(n>m);$$

Presented to the Society, September 2, 1949; received by the editors August 1, 1949.

[1] S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932.

[2] F. Hausdorff, Math. Zeit, vol. 9 (1921) pp. 74–109, 280–299.

[3] The domain of uniqueness of the ordinary Banach limits has been determined by G. G. Lorentz, Acta Math. vol. 80 (1948) pp. 167–190.

and the $\alpha(u)$ are real functions of bounded variation (BV) in the interval $(0, 1)$ satisfying the end conditions: $\alpha(0)=\alpha(0+)=0$, $\alpha(1)=1$.

H in $\mathfrak{H}$ is termed *definite* or *completely regular*[4] (c.r.) if the generating function $\alpha(u)$ is non-decreasing or, equivalently, if all the matrix elements a_{mn} are non-negative. If H is c.r., $\|H\|=1$ and

$$L_*(x) \leqq L_*(Hx) \leqq L^*(Hx) \leqq L^*(x). \tag{3}$$

The standard decomposition of BV functions yields the canonical resolution of a regular H:

$$H = aH_1 - bH_2 \qquad (H_1, H_2 \text{ c.r.}; a - b = 1, b \geqq 0). \tag{4}$$

Henceforth we need consider only the convex semi-group $\mathfrak{H}_+ \subset \mathfrak{H}$ of c.r. Hausdorff transformations and may replace (V) with the equivalent condition

(V bis) $L(Hx) = L(x)$ $(H \in \mathfrak{H}_+)$.

The following conventions will materially shorten our work:

DEFINITION 1. *C is the class of convergent sequences; C_0 the subclass of sequences converging to 0 (null sequences). $x \sim y$ if and only if $x-y \in C_0$.*

Clearly $\sim$ is a congruence relation under addition, scalar multiplication, and the operations H.

DEFINITION 2. $P_+(x) = \inf_{H \in \mathfrak{H}_+} L^*(Hx)$; $P_-(x) = -P_+(-x) = \sup_{H \in \mathfrak{H}_+} L_*(Hx)$.

Our basic results may now be formulated as

THEOREM 1. (a) *A n.a.s.c. that the linear functional $L(x)$ on m be a B-H functional is that*

$$P_-(x) \leqq L(x) \leqq P_+(x) \qquad (x \in m). \tag{5}$$

(b) *B-H functionals exist.*

(c) *A n.a.s.c. that all B-H functionals coincide at x is that $P_-(x) = P_+(x)$.*

That B-H functionals must satisfy the inequality (5) is an immediate consequence of (1) and (V bis). The condition $P_-(x)=P_+(x)$ is then obviously sufficient to ensure coincidence at x.

3. Convexity. To establish the remaining assertions of Theorem 1, we derive the convexity and other necessary properties of $P_+(x)$, and then adapt the Hahn-Banach[1] extension procedure. The decisive role of the Hölder-Cesàro transformation $H_0x=y$, where $y_n=(n+1)^{-1}$

[4] Cf. G. H. Hardy, *Divergent series*, Oxford, 1949.

$\cdot(x_0+\cdots+x_n)$ corresponds to $\alpha(u)=u$, is due to the following elementary property:

LEMMA 1. $H_0Sx\sim H_0x\sim SH_0x$.

The requisite properties of $P_\pm(x)$ are

LEMMA 2. (a) $P_+(x+y)\leqq P_+(x)+P_+(y)$;
(b) $P_+(ax)=aP_+(x)$ $(a\geqq 0)$;
(c) $P_-(x)\leqq P_+(x)$;
(d) $|P_\pm(x)|\leqq\|x\|$; $P_\pm(x)\geqq 0$ *if* $x\geqq 0$;
(e) *If* $x\in C$, $P_\pm(x)=\lim_n x_n$ *and* $P_\pm(x+y)=lim_n x_n+P_\pm(y)$; *in particular*, $P_\pm(x)=P_\pm(y)$ *if* $x\sim y$;
(f) $P_\pm(Hx)=P_\pm(x)$ $(H\in\mathfrak{H}_+)$;
(g) $P_\pm(Sx)=P_\pm(x)$;
(h) $P_\pm(Sx-x)=0$;
(i) $P_\pm(Hx-x)=0$ $(H\in\mathfrak{H}_+)$.

PROOF. (a) Given $\epsilon>0$, there exist H_1, H_2 in $\mathfrak{H}_+$ such that $L^*(H_1x)\leqq P_+(x)+\epsilon$, $L^*(H_2y)\leqq P_+(y)+\epsilon$. Set $H_3=H_2H_1=H_1H_2$. Then $L^*(H_3x)=L^*(H_2H_1x)\leqq L^*(H_1x)\leqq P_+(x)+\epsilon$. Similarly $L^*(H_3y)\leqq P_+(y)+\epsilon$. Hence $P_+(x+y)\leqq L^*(H_3(x+y))\leqq L^*(H_3x)+L^*(H_3y)\leqq P_+(x)+P_+(y)+2\epsilon$. Since $\epsilon>0$ was arbitrary, $P_+(x+y)\leqq P_+(x)+P_+(y)$.

(b), (d), and (e) are obvious, and (c) follows on setting $y=-x$ in (a). (f) is an easy consequence of the definition of $P_\pm(x)$ by way of (3) and the semi-group property of $\mathfrak{H}_+$. Lemma 1 and (e), (f) now yield

(g) $P_\pm(Sx)=P_\pm(H_0Sx)=P_\pm(H_0x)=P_\pm(x)$;
(h) $P_\pm(Sx-x)=P_\pm(H_0Sx-H_0x)=0$.

To establish (i) note the ergodic[5] property $\|H_nH-H_n\|\leqq 2(n+1)^{-1}$ of the operators $H_n=(n+1)^{-1}\sum_0^n H^i\in\mathfrak{H}_+$. Thus $|P_\pm(Hx-x)|=|P_\pm((H_nH-H_n)x)|\leqq\|(H_nH-H_n)x\|\leqq 2(n+1)^{-1}\|x\|$. n being an arbitrary integer, $P_\pm(Hx-x)=0$.

We complete the proof of Theorem 1. That any linear L satisfying (5) is a B-H functional is now immediate: (II) follows from the identity $P_\pm(1)=1$; (III) from Lemma 2(d); the identities $L(Sx-x)=0$ (IV) and $L(Hx-x)=0$ (V bis) from Lemma 2, (h) and (i) respectively.

Application of the Hahn-Banach extension theorem to the convex function $P_+(x)$ yields linear functionals L such that $L(x)\leqq P_+(x)$. But then $L(x)=-L(-x)\geqq -P_+(-x)=P_-(x)$, whence the L satisfy (5).

[5] Cf. W. F. Eberlein, Proc. Nat. Acad. Sci. U.S.A. vol. 34 (1948) pp. 43–47; Trans. Amer. Math. Soc. vol. 67 (1949) pp. 217–240.

Finally, the necessity of the condition $P_-(x)=P_+(x)$ for coincidence of all B-H functionals at x is implicit in the extension procedure: If $P_-(y)<P_+(y)$ for some y in m, the value of $L(y)$ can be chosen arbitrarily in the interval $P_-(y)\leqq L(y)\leqq P_+(y)$. For let $L(x)=\lim_n x_n$ $(x\in C)$, and recall that we may continue L into the subspace $x+ay$ $(x\in C, a$ real) by choosing $L(y)$ arbitrarily in the interval

$$\sup_{x\in C}\{P_-(x+y)-L(x)\}\leqq L(y)\leqq \inf_{x\in C}\{P_+(x+y)-L(x)\}.$$

But the second inequality reduces to the first (Lemma 2(e)), and our assertion follows on continuing L into the whole space m.

Further developments will appear in a second note.

UNIVERSITY OF WISCONSIN

PROOF OF A THEOREM OF JACOBI

N. J. FINE

Jacobi[1] proved the following theorem:
If $G(z)$ is defined in $[-1, 1]$, then

$$(1)\qquad I_n \equiv \int_0^\pi G(\cos x)\cos nx\,dx = \frac{1}{1\cdot 3\cdot 5\cdots(2n-1)}\int_0^\pi G^{(n)}(\cos x)\sin^{2n}x\,dx.$$

His first proof, for the case in which $G(z)$ may be expanded in a power series, depends on the formula

$$(2)\qquad \int_0^\pi \cos^p x\cos nx\,dx = \frac{p(p-1)\cdots(p-n+1)}{1\cdot 3\cdot 5\cdots(2n-1)}\int_0^\pi \cos^{p-n}x\sin^{2n}x\,dx,$$

which is itself a special case of (1). His second proof, which assumes nothing about the derivatives of $G(z)$ of order exceeding n, depends on the lemma

$$(3)\qquad \frac{d^{n-1}}{dz^{n-1}}(1-z^2)^{(2n-1)/2} = (-1)^{n-1}1\cdot 3\cdot 5\cdots(2n-1)\frac{\sin nx}{n},$$

where $z=\cos x$. He points out that (3) may also be deduced from (1).

We offer here a short proof by induction which does not involve previous knowledge of (2) or (3). For $n=1$ the theorem is seen to be true by an integration by parts. Now

$$I_{n+1} = \int_0^\pi G(\cos x)\cos x\cos nx\,dx - \int_0^\pi G(\cos x)\sin x\sin nx\,dx$$

$$= \int_0^\pi G(\cos x)\cos x\cos nx\,dx - n\int_0^\pi G_1(\cos x)\cos nx\,dx$$

by integration by parts, where $G_1(z)$ is an integral of $G(z)$. Applying the induction hypothesis to $F(z)=zG(z)-nG_1(z)$ and observing that $F^{(n)}(z)=zG^{(n)}(z)$, we get

Presented to the Society, February 25, 1950; received by the editors August 16, 1949.

[1] C. G. J. Jacobi, J. Reine Angew. Math. vol. 15 (1836) pp. 1–26.

$$I_{n+1} = \int_0^\pi F(\cos x) \cos nx\, dx$$

$$= \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \int_0^\pi G^{(n)}(\cos x) \sin^{2n} x \cos x\, dx.$$

Another integration by parts yields

$$I_{n+1} = \frac{1}{1 \cdot 3 \cdot 5 \cdots (2n+1)} \int_0^\pi G^{(n+1)}(\cos x) \sin^{2n+2} x\, dx,$$

and the theorem is proved.

UNIVERSITY OF PENNSYLVANIA

A REMARK ON THE MODULI OF RIEMANN SURFACES OF GENUS 2

P. R. GARABEDIAN[1]

1. **Introduction.** The object of this paper is to show how the problem of determining moduli of conformal type for certain Riemann surfaces can be solved by using the theory of Riemann theta-functions. It is to be hoped that the basic ideas presented here, together with the results of Schottky [6][2] and Poincaré [5] on relations between periods of integrals of the first kind on closed Riemann surfaces, will lead to further work with the theta-function method on the problem of determining conformal moduli.

Our principal theorem is an extension to closed surfaces of genus 2 of work done by Ahlfors and Beurling [1] on the moduli of triply-connected regions. Their result is found to be a special case of the theorem for closed surfaces, as will be seen from a consideration of the double of the triply-connected region, which is a symmetric closed surface of genus 2. The method of theta-functions used in this paper is, of course, quite different from the method of extremal length developed by Ahlfors and Beurling. Relations between the two methods and effective extensions of both to the treatment of moduli of more general Riemann surfaces offer an interesting field for further study. We remark that results on infinitesimally close Riemann surfaces have been obtained already in the general case by use of variational methods [2].

2. **Riemann theta-functions.** We consider a closed Riemann surface S of genus 2. On S we introduce the usual system of canonical cuts α_1, α_2, β_1, β_2, and we define two linearly independent normal integrals of the first kind, $w_1(z)$ and $w_2(z)$, for z in S. The integral $w_\mu(z)$ has about α_1 and α_2 periods $\delta_{\mu 1}\pi i$ and $\delta_{\mu 2}\pi i$, and has about β_1 and β_2 periods $a_{\mu 1}$ and $a_{\mu 2}$, $\mu=1, 2$, where $\delta_{\mu\nu}$ is the Kronecker delta. The quadratic form

$$\sum_{\mu,\nu=1}^{2} \operatorname{Re}\{a_{\mu\nu}\} x_\mu x_\nu$$

is negative definite, when x_1 and x_2 are real. Also, we have the symmetry relation $a_{\mu\nu}=a_{\nu\mu}$.

Presented to the Society, October 29, 1949; received by the editors July 25, 1949.

[1] National Research Fellow.

[2] Numbers in brackets refer to the bibliography at the end of the paper.

We define the Riemann theta-function

$$(1) \qquad \theta(u_1, u_2) = \sum_{n_1, n_2 = -\infty}^{\infty} \exp\left\{\sum_{\mu,\nu=1}^{2} a_{\mu\nu} n_\mu n_\nu + 2\sum_{\mu=1}^{2} u_\mu n_\mu\right\}$$

associated with the closed Riemann surface S. This series converges for all complex values of u_1, u_2, and it represents an analytic function of considerable importance for the consideration of functions on S. Riemann found, using the theory of this theta-function, that every pair of complex numbers V_1, V_2 has a representation

$$(2) \qquad V_1 = w_1(z_1) + w_1(z_2), \qquad V_2 = w_2(z_1) + w_2(z_2),$$

where z_1 and z_2 are points on S, and he showed that this representation is in general unique (Jacobi inversion problem). Furthermore, he proved the following representation, which is of fundamental significance for our work and is connected with the uniqueness of the representation (2). If the additive constants implicitly contained in the definition of the integrals of the first kind $w_1(z)$ and $w_2(z)$ on S are suitably chosen, then the zeros V_1, V_2 of the function $\theta(u_1, u_2)$, that is, the points V_1, V_2 with

$$\theta(V_1, V_2) = 0,$$

are given by

$$(3) \qquad V_1 = w_1(z), \qquad V_2 = w_2(z),$$

where z is a variable point of S, and this representation is in general unique. We shall consider throughout this paper that the constants implicitly contained in the integrals $w_1(z)$ and $w_2(z)$ have been chosen so that (3) holds.

A detailed presentation of the notions and results which we have described here is given by Neumann [4]. We remark that an inversion formula of the form (2) completely solves the problem of finding moduli for the closed Riemann surfaces of genus 1. Indeed, such a formula leads to the complete discussion of these surfaces in terms of elliptic functions. Our principal object now will be to use the more refined formula (3) in order to find moduli for the closed Riemann surfaces S of genus 2.

3. **Moduli of conformal type.** The Riemann surface S which we have discussed depends in its conformal type upon 6 real parameters, which we call a set of conformal moduli of S. That is, with each such surface we can associate an ordered set of 6 real numbers in such a way that there is a conformal mapping of preassigned topological

character of any one such surface, S, upon any other, S^*, if and only if the corresponding sets of 6 numbers are identical. We shall prove in this section that the real and imaginary parts of the three complex numbers a_{11}, a_{13}, and a_{22} yield a set of conformal moduli of S of the kind we describe. By virtue of the identity $a_{12}=a_{21}$, we see that this will be proved if we show that the matrix $((a_{\mu\nu}))$ determines the conformal type of S uniquely.

Let S^* be a second closed Riemann surface of genus 2, and associate with S^* a system of canonical cuts α_1^*, α_2^*, β_1^*, β_2^* and a corresponding pair of normalized integrals of the first kind $w_1^*(z^*)$, $w_2^*(z^*)$. Also to be associated with S^* is a matrix of periods $((a_{\mu\nu}^*))$ of the integrals $w_1^*(z^*)$ and $w_2^*(z^*)$ and a Riemann theta-function $\theta^*(u_1, u_2)$. We shall show that if the matrices $((a_{\mu\nu}))$ and $((a_{\mu\nu}^*))$ are identical, that is, if $a_{11}=a_{11}^*$, $a_{12}=a_{12}^*$, and $a_{22}=a_{22}^*$, then S can be mapped conformally and one-one upon S^*, and the mapping which we shall construct in the proof will send the set of canonical cuts α_1, α_2, β_1, β_2 into a set of canonical cuts on S^* which are equivalent topologically to α_1^*, α_2^*, β_1^*, β_2^*.

Indeed, suppose that $((a_{\mu\nu}))$ and $((a_{\mu\nu}^*))$ are identical. Then from the definition (1) of the theta-function $\theta(u_1, u_2)$ and the corresponding definition of $\theta^*(u_1, u_2)$, we obtain the identity

$$\theta(u_1, u_2) \equiv \theta^*(u_1, u_2). \tag{4}$$

Hence the manifold of zeros V_1, V_2 of $\theta(u_1, u_2)$ is the same as the manifold of zeros V_1, V_2 of $\theta^*(u_1, u_2)$. Thus we have not only the representation (3) of each point V_1, V_2 on this manifold, but also the representation

$$V_1 = w_1^*(z^*), \qquad V_2 = w_2^*(z^*), \tag{5}$$

where z^* is a variable point on S^*. Thus we obtain the relations

$$w_1(z) = w_1^*(z^*), \qquad w_2(z) = w_2^*(z^*), \tag{6}$$

between suitable points z on S and z^* and S^*.

We maintain that the correspondence (6), which is in general one-one, yields a conformal mapping $z^*(z)$ of S upon S^* and at the same time yields the inverse mapping $z(z^*)$ of S^* upon S. These mappings can, indeed, be obtained by solving the first implicit equation (6) first for z^* and then for z. The solutions can break down only at the critical points of $w_1^*(z^*)$ on S^* and the critical points of $w_1(z)$ on S. But near these points, we can use the second implicit equation (6) in order to determine the mappings $z^*(z)$ and $z(z^*)$. Indeed, the integrals $w_1(z)$ and $w_2(z)$ cannot have a critical point in common, nor can

the integrals $w_1^*(z^*)$ and $w_2^*(z^*)$.

To prove the last statement, say for the pair of integrals $w_1(z)$ and $w_2(z)$, we remark that by the Riemann-Roch theorem the functions $w_1(z)$ and $w_2(z)$ each have precisely two critical points on S. Thus if $w_1(z)$ and $w_2(z)$ had a critical point in common, then the single-valued analytic function

$$\frac{dw_1(z)}{dw_2(z)}$$

would have either one pole or no poles on S, and hence it would be either univalent or constant on S. But both these possibilities are excluded by the fact that S has genus 2, and thus we should arrive at a contradiction.

Coming back once more to the equations (6), we see that their solutions $z^*(z)$ and $z(z^*)$ on S and S^* are single-valued functions, with values in S^* and S, since the period matrices $((a_{\mu\nu}))$ and $((a^*_{\mu\nu}))$ are identical. Thus, indeed, we obtain a one-one conformal mapping of S upon S^*, as desired. It follows that the matrix $((a_{\mu\nu}))$ yields a set of 6 real moduli for S, as stated above.[3]

4. Symmetric cases. The result of the previous section can be used to obtain a theorem of the type investigated by Ahlfors and Beurling [1] for triply-connected regions. Let D be a domain of the complex z-plane bounded by three curves C_1, C_2, and C_3, which we may assume with no loss of generality to be analytic. Let $\omega_\nu(z)$ be that harmonic function in D with boundary values 1 on C_ν and zero boundary values on the C_μ with $\mu \neq \nu$, $\nu = 1, 2, 3$. Let $W_\nu(z)$ be the corresponding analytic function in D with $\omega_\nu(z) = \mathrm{Im}\,\{W_\nu(z)\}$. The three functions $W_\nu(z)$ have in D a matrix of real periods, $((p_{\mu\nu}))$, and these periods are completely determined by the subset p_{11}, p_{12}, and p_{22}. We maintain that p_{11}, p_{12}, and p_{22} are a set of three conformal moduli for the triply-connected domain D, and we shall prove this using the result of §3, although simpler, elementary proofs are available.

We can associate with D a closed Riemann surface S of genus 2, known as the double of D. The surface S is obtained by taking two replicas of D and identifying corresponding boundary points. The

[3] This procedure can also be used to show that there is always a nontrivial conformal mapping of S onto itself. Indeed, since $\theta(u_1, u_2)$ is an odd function, we have in addition to (3) the representation $V_1 = -w_1(z)$, $V_2 = -w_2(z)$ of the zeros of $\theta(u_1, u_2)$. Thus we can define a conformal mapping $\bar{z}(z)$ of S onto itself by means of the relations $w_1(\bar{z}) = -w_1(z)$, $w_2(\bar{z}) = -w_2(z)$.

normal integrals of the first kind $w_1(z)$ and $w_2(z)$ on S can be chosen so that

$$w_\mu(z) = \frac{\pi}{2} W_\mu(z) + \text{const.}, \qquad \mu = 1, 2, \tag{7}$$

in D. Thus the matrix $((a_{\mu\nu}))$ for S is given by the relations

$$2a_{11} = \pi p_{11}, \qquad 2a_{12} = \pi p_{12}, \qquad 2a_{22} = \pi p_{22}. \tag{8}$$

We see immediately that if the periods p_{11}, p_{12}, p_{22} associated with the domain D are identical with the periods p_{11}^*, p_{12}^*, p_{22}^* associated with a second triply-connected plane domain D^*, then the double S of D can be mapped conformally upon the double S^* of D^*, by §3.

Let $z^*(z)$ be the mapping of S upon S^*. Then we have from (6) and (7), for a suitably chosen real constant λ, the relation

$$\omega_1^*(z^*(z)) = \omega_1(z) + \lambda, \tag{9}$$

where $\omega_\mu^*(z^*)$, $\mu = 1, 2, 3$, are the harmonic functions in D^* corresponding to the $\omega_\mu(z)$ in D, with boundary values 1 or 0 on the respective boundary components of D. Now the critical points of $\omega_1^*(z^*)$ and $\omega_1(z)$ must correspond in the mapping $z^*(z)$, since this mapping is single-valued and univalent. Let z_0 and z_0^* be a pair of corresponding critical points, and let z_1 and z_1^* be the points symmetric to z_0 and z_0^* on S and S^*. Of course, z_1 and z_1^* are critical points of $w_1(z)$ and $w_1^*(z^*)$, respectively, by symmetry.

We can continue the identity (9) analytically from z_0 to z_1, across C_2. We have thus on the one hand

$$\omega_1^*(z^*(z_0)) = \omega_1(z_0) + \lambda, \tag{10}$$

while on the other hand, letting $z = z_1$ in (9), we obtain

$$-\omega_1^*(z^*(z_0)) = -\omega_1(z_0) + \lambda, \tag{11}$$

since $\omega_1(z_1) = -\omega_1(z_0)$, $\omega_1^*(z_1^*) = -\omega_1^*(z_0^*)$. Adding (10) and (11), we have

$$2\lambda = 0,$$

and thus (9) takes the simpler form

$$\omega_1^*(z^*(z)) = \omega_1(z). \tag{12}$$

It is clear from (12) that the conformal mapping $z^*(z)$ of S upon S^* yields, in particular, a conformal mapping of D upon D^*. The desired theorem on the conformal moduli of triply-connected domains follows as an immediate consequence, since this proves that p_{11},

p_{12}, and p_{22} are a set of conformal moduli for D.

We remark, finally, that in a similar way we can find a set of three conformal moduli for the Riemann surfaces R of genus 1 with 1 boundary curve. Here again, the construction depends upon a consideration of the symmetric double S of R, obtained by identifying boundary points of two replicas of R. Clearly, the double S of R has genus 2, and the period matrix $((a_{\mu\nu}))$ associated with S satisfies three independent symmetry relations, namely, $a_{11}=\bar{a}_{22}$, $a_{12}=\bar{a}_{21}$, so that we are left with 3 real conformal moduli for R. The detailed investigation of this second symmetric case is left to the reader, and its discussion will lead to a more complete understanding of the present method. Suffice it to say that here the role which the harmonic functions $\omega_\mu(z)$ played for D is played by the integrals over the two canonical cuts on R of the normal derivative of the Green's function of R.

Bibliography

1. L. Ahlfors and A. Beurling, *Invariants conformes et problèmes extrémaux* II, Comptes Rendus du Dixième Congrès des Mathématiciens Scandinaves, Copenhagen, 1947, pp. 341–351.

2. P. R. Garabedian and M. Schiffer, *Identities in the theory of conformal mapping*, Trans. Amer. Math. Soc. vol. 65 (1949) pp. 187–238.

3. A. Krazer, *Lehrbuch der Thetafunktionen*, Leipzig, 1903.

4. C. Neumann, *Vorlesungen über Riemanns Theorie der Abel'schen Integrale*, Leipzig, 1884.

5. H. Poincaré, *Remarques diverses sur les fonctions abéliennes*, J. Math. Pures Appl. (5) vol. 1 (1895) pp. 219–314.

6. F. Schottky, *Zur Theorie der Abelschen Functionen von vier Variabeln*, J. Reine Angew. Math. vol. 102 (1888) pp. 304–352.

Stanford University

THE REPRESENTATION OF REAL NUMBERS

O. W. RECHARD

1.0. **Introduction.** Let $p \geqq 2$ be a positive integer and denote by E_p the class of all continuous, strictly increasing functions $f(x)$ on the interval $0 \leqq x \leqq p$ with $f(0) = 0$ and $f(p) = 1$. In a generalization of the decimal representation, Everett[1] has associated with every real number $0 \leqq \gamma < 1$ a sequence of integers $c_1, c_2, \cdots$, with $0 \leqq c_i \leqq p-1$ by means of the following algorithm:

$$\text{(A)} \qquad \begin{aligned} \gamma &= f(\gamma_1), & 0 \leqq \gamma_1 < p, \\ \gamma_1 &= c_1 + f(\gamma_2), & 0 \leqq \gamma_2 < p, \\ \gamma_2 &= c_2 + f(\gamma_3), & 0 \leqq \gamma_3 < p, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

where $f(x)$ is an arbitrary function in the class E_p.

Some functions (for example, $f(x) = x/p$, which leads to the representation of a number as a decimal to the base p) when employed in the algorithm (A) yield one-one correspondences between real numbers and sequences of integers mod p. On the other hand, any function, for example, whose graph has more than one point in common with any of the straight line segments connecting the points $(j, 0)$ and $(j+1, 1)$, $j = 0, 1, \cdots, p-1$, will obviously lead to a correspondence which is many-one. We shall denote by E_p^* the subclass of E_p consisting of those functions which in the algorithm (A) give rise to one-one correspondences.

The present paper contains very simple characterizations of those correspondences between real numbers and sequences of integers mod p which can be obtained by applying the algorithm (A) with functions from the classes E_p^* and $E_p - E_p^*$ respectively. By means of these characterizations it is possible to settle two of the problems raised by Everett and to give an answer (albeit not a completely satisfactory one) to a third, namely that of characterizing the class E_p^* itself.

2.0. **Associated functions.** For every function $f(x)$ in E_p we can define an *associated function* $F(x)$ on $0 \leqq x \leqq 1$ as follows. Define $F(1) = 1$. If $0 \leqq \gamma < 1$, let $c_1, c_2, \cdots$ be the sequence of integers asso-

Presented to the Society, November 26, 1948; received by the editors August 12, 1949.

[1] *Representations for real numbers*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 861–869.

ciated with γ by the algorithm (A) utilizing the function $f(x)$; then define $F(\gamma) = \sum_{j=1}^{\infty} c_j/p^j$.

2.1. THEOREM 1. *If $f(x)$ is any function in the class E_p, its associated function $F(x)$ is continuous and nondecreasing on $0 \leqq x \leqq 1$.*

PROOF. Note first that for $0 \leqq \gamma < 1$, $F(\gamma) \leqq \sum_{j=1}^{\infty} p-1/p^j = 1 = F(1)$. Now fix any two points α and β such that $0 \leqq \alpha < \beta < 1$, and let the sequences obtained from applying (A) to α and β be respectively $a_1, a_2, \cdots$, and $b_1, b_2, \cdots$. If $F(\alpha) \neq F(\beta)$, let n be the first integer such that $a_n \neq b_n$. At the $(n+1)$st stage in applying (A) to α and β we have respectively

$$\begin{aligned} \alpha_n &= a_n + f(\alpha_{n+1}), \\ \beta_n &= b_n + f(\beta_{n+1}), \end{aligned} \qquad f(\alpha_{n+1}), f(\beta_{n+1}) < 1,$$

and since the monotonicity of $f(x)$ implies $\alpha_n < \beta_n$ it follows that $a_n < b_n$. Consequently, $F(\alpha) < F(\beta)$, and $F(x)$ is nondecreasing on $0 \leqq x \leqq 1$.

Clearly, the range of values of $F(x)$ includes every terminating decimal on $[0, 1]$. In fact, if $c_1, c_2, \cdots, c_n$ is any finite sequence of integers mod p, then

$$F[f(c_1 + f(c_2 + \cdots + f(c_n) \cdots))] = 0.c_1c_2 \cdots c_n.$$

Consequently, since the terminating decimals are dense on $[0, 1]$ and $F(x)$ is monotonic, it follows that $F(x)$ is continuous and, in fact, takes on every value on $[0, 1]$.

2.2. Note that for $f(x) \in E_p$ its associated function is uniquely defined by the functional equation

$$\text{(B)} \qquad F[f(x)] = [x]/p + F(x - [x])/p, \qquad 0 \leqq x \leqq p,$$

where $[x]$ denotes the largest integer in x. Certainly, the associated function for $f(x)$ satisfies (B). Assume that a function $F(x)$ defined on $[0, 1]$ satisfies (B); then $F(0) = 0$ and $F(1) = 1$. Moreover, if under the algorithm (A), $\gamma < 1$ corresponds to the sequence $c_1, c_2, \cdots$, then we have

$$\begin{aligned} \gamma &= f(\gamma_1), & 0 \leqq \gamma_1 < p, \\ \gamma_1 &= c_1 + f(\gamma_2), & 0 \leqq \gamma_2 < p, \\ \gamma_2 &= c_2 + f(\gamma_3), & 0 \leqq \gamma_3 < p, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

and

$$F(\gamma) = F[f(\gamma_1)] = c_1/p + F[f(\gamma_2)]/p$$
$$= c_1/p + c_2/p^2 + F[f(\gamma_3)]/p^2 = \cdots .$$

2.3. If $F(x_1) = F(x_2) = c$ for $x_1 < x_2$ (and, hence, $F(x) = c$ for $x_1 \leqq x \leqq x_2$), then c is called a *level value* for $F(x)$. It is easy to see that $F(x)$ has the following property: if the algorithm (A) when applied to two distinct numbers α and $\bar{\alpha}$ yields in each case the sequence $a_1, a_2, \cdots$, so that $0.a_1a_2\cdots$ is a level value for $F(x)$, then each of the numbers $0.a_2a_3\cdots$ and $0.ia_1a_2\cdots$, $i = 0, 1, \cdots, p-1$, must also be a level value. For if

$$\begin{aligned} \alpha &= f(\alpha_1), & 0 \leqq \alpha_1 < p, \\ \alpha_1 &= a_1 + f(\alpha_2), & 0 \leqq \alpha_2 < p, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

and

$$\begin{aligned} \bar{\alpha} &= f(\bar{\alpha}_1), & 0 \leqq \bar{\alpha}_1 < p, \\ \bar{\alpha}_1 &= a_1 + f(\bar{\alpha}_2), & 0 \leqq \bar{\alpha}_2 < p, \\ &\cdots\cdots\cdots\cdots\cdots\cdots , \end{aligned}$$

then the two distinct numbers $\alpha_1 - a_1$ and $\bar{\alpha}_1 - a_1$ each yield under (A) the sequence $a_2, a_3, \cdots$. Similarly, the two numbers $f(\alpha+i)$ and $f(\bar{\alpha}+i)$ each yield the sequence $i, a_1, a_2, \cdots$.

This property of the function $F(x)$ implies that it is either strictly increasing (as is the case if and only if $f(x)$ is in E_p^*) or else its level values, and hence its intervals of constancy, are dense on $[0, 1]$.

3.0. **The one-one case.** If $f(x) \in E_p^*$, its associated function is continuous and strictly increasing on $[0, 1]$ to $[0, 1]$. We next inquire whether or not every such function is associated with some function in E_p^*.

THEOREM 2. *If $F(x)$ is any continuous, strictly increasing function on $0 \leqq x \leqq 1$ with $F(0) = 0$ and $F(1) = 1$, then it is the associated function for a unique function $f(x)$ in the class E_p^*.*

PROOF. Define $f(x) = F^{-1}[[x]/p + F(x - [x])/p]$, $0 \leqq x \leqq p$. Clearly, $f(x) \in E_p$, and since $F(x)$ satisfies the functional equation (B), it must be the associated function for $f(x)$. This, in turn, implies that $f(x) \in E_p^*$. Since any function in E_p^* which has $F(x)$ as its associated function must satisfy the above equation, $f(x)$ is unique.

3.1. Theorems 1 and 2 completely characterize those one-one correspondences between real numbers and sequences of integers mod p which can be obtained by applying the algorithm (A) with

functions from the class E_p^*. *They are simply the correspondences which can be obtained from the class of all continuous, strictly increasing functions* $F(x)$ *on* $[0, 1]$ *to* $[0, 1]$ *by defining the sequence corresponding to the number* γ *to be the sequence of integers in the non-*$(p-1)$*-terminating expansion of* $F(\gamma)$ *as a decimal to the base* p.

3.2. Also as a result of Theorems 1 and 2 we can state that *a necessary and sufficient condition for a function* $f(x)$ *to belong to the class* E_p^* *is that it have the form* $f(x)=F^{-1}[[x]/p+F(x-[x])/p]$, $0\leqq x\leqq p$, *where* $F(x)$ *is a continuous, strictly increasing function on* $[0, 1]$ *to* $[0, 1]$.

As an example, if $p=2$ and $F(x)=x^n$, then

$$f(x)=\begin{cases} x/2^{1/n}, & 0\leqq x\leqq 1,\\ [(x-1)^n+1]^{1/n}/2^{1/n}, & 1\leqq x\leqq 2.\end{cases}$$

From this we can give a periodic representation of the nth root of a rational number $R<1$. In fact, $R^{1/n}=f(r_1+f(r_2+\cdots))$, where $0.r_1r_2\cdots$ is the dyadic expansion of R.

3.3. Let A be any denumerable dense set on $[0, 1]$ with $0\in A$ and $1\in A$ (for example, the set of all algebraic numbers on $[0, 1]$). *There exists a function* $f(x)\in E_p^*$ *such that a number* α *is in* A *if and only if* $\alpha=f(a_1+f(a_2+\cdots+f(a_n)\cdots))$ *for some finite sequence of integers* $a_1, a_2, \cdots, a_n$ *with* $0\leqq a_j\leqq p-1$. For, by virtue of Theorem 2, it is sufficient to show that there exists a continuous, strictly increasing function $F(x)$ on $[0, 1]$ to $[0, 1]$ which maps the set A onto the entire set T of numbers on $[0, 1]$ whose decimal expansions to the base p are terminating.

To define the function $F(x)$, order the sets $A: a_0=0, a_1=1, a_2, \cdots$, and $T: t_0=0, t_1=1, t_2, \cdots$. Define $F(0)=0$, $F(1)=1$, and $F(a_2)=t_2$. Now suppose that $F(x)$ has been defined in monotonically increasing fashion at all points a_j, $j<n$, with values in the set T. Let a_i be the largest number a_j such that $j<n$ and $a_j<a_n$, and let a_k be the smallest number a_j such that $j<n$ and $a_j>a_n$. Then define $F(a_n)$ to be the first number t in T such that $F(a_i)<t<F(a_k)$. Clearly, $F(x)$ is thus defined on the set A and is strictly increasing. Moreover, on A, $F(x)$ takes on every value in the set T. Since both T and A are dense in $[0, 1]$, we can extend $F(x)$ continuously to the entire interval $[0, 1]$ and obtain the desired function.

4.0. **The many-one case.** Let us turn now to the problem of characterizing those many-one correspondences between real numbers and sequences of integers mod p which can be obtained by applying the algorithm (A) with functions from the class $E_p-E_p^*$. In terms of

associated functions, the problem is that of characterizing the functions $F(x)$ which are associated with functions $f(x)$ in $E_p - E_p^*$.

4.1. A continuous, nondecreasing function $F(x)$ on $[0, 1]$ with $F(0)=0$ and $F(1)=1$ will be said to have the property (P) if, whenever a number $0.a_1a_2\cdots$ (decimal to the base p) is a level value for $F(x)$, each of the numbers $0.a_2a_3\cdots$ and $0.ia_1a_2\cdots$, $i=0, 1, \cdots, p-1$, is also a level value for $F(x)$. It shall be understood that if a number which can be represented by both a terminating and a $(p-1)$-terminating decimal is a level value, then every such number between 0 and 1, and one, but not necessarily both, of the numbers 0 and 1, must also be level values.

If $G(x)$ is defined for $0 \leqq x \leqq p$ by $G(x) = [x]/p + F(x-[x])/p$, and if $F(x)$ has the property (P), then a number t is a level value for $F(x)$ if and only if it is a level value for $G(x)$. For suppose[2] $t = 0.a_1a_2\cdots$ is a level value for $F(x)$; then property (P) implies that $F(x_1) = F(x_2) = tp - a_1 = 0.a_2a_3\cdots$ for $0 < x_1 < x_2 < 1$. Consequently, $G(x_1 + a_1) = a_1/p + F(x_1)/p = a_1/p + F(x_2)/p = G(x_2 + a_1) = t$, so t is a level value for $G(x)$. Conversely, if $G(\xi_1) = G(\xi_2) = t$ for $\xi_1 \neq \xi_2$ and $[\xi_1] = [\xi_2]$, then $F(\xi_1 - [\xi_1]) = F(\xi_2 - [\xi_2]) = pt - [\xi_1] = 0.c_1c_2\cdots$ and property (P) implies that $t = 0.[\xi_1]c_1c_2\cdots$ is a level value for $F(x)$.

4.2. In §2.3 we saw that the associated function of every function in the class $E_p - E_p^*$ is nondecreasing but not strictly increasing on $[0, 1]$ to $[0, 1]$ and has the property (P). We shall now show that the converse is also true.

THEOREM 3. *Let $F(x)$ be a nondecreasing function on $[0, 1]$ with $F(0)=0$ and $F(1)=1$. If $F(x)$ is not strictly increasing and has the property* (P), *then it is the associated function for infinitely many functions $f(x)$ in the class $E_p - E_p^*$.*

PROOF. As in §4.1, define $G(x) = [x]/p + F(x-[x])/p$ for $0 \leqq x \leqq p$; then we seek a function $f(x)$ in E_p such that $F[f(x)] = G(x)$.

For every number t, $0 \leqq t \leqq 1$, denote by $E_F(t)$ the set (either an interval or a single point) of points x such that $F(x) = t$. Define $m_F(t) = \text{g.l.b. } E_F(t)$ and $M_F(t) = \text{l.u.b. } E_F(t)$. Similarly, denote by $E_G(t)$ the set of points x such that $G(x) = t$, and define $m_G(t) = \text{g.l.b. } E_G(t)$ and $M_G(t) = \text{l.u.b. } E_G(t)$. Now define $f[m_G(t)] = m_F(t)$ and $f[M_G(t)] = M_F(t)$. Then $f(x)$ is defined for every point on $[0, p]$ not in the interior of an interval of constancy of $G(x)$. The fact that $F(x)$ and $G(x)$ are nondecreasing and that t is a level value for $F(x)$ if and only if it is a level value for $G(x)$ insures that $f(x)$ is strictly increasing.

[2] If $t = q/p$ we take the terminating or nonterminating decimal representation for t according as 0 or 1 is a level value for $F(x)$.

On every open interval of constancy of $G(x)$, that is, on the intervals $m_G(t) < x < M_G(t)$, we can define $f(x)$ quite arbitrarily subject only to the restrictions that it must be strictly increasing and must take on every value between $m_F(t)$ and $M_F(t)$. The function $f(x)$ is then defined on the entire interval $[0, p]$, is strictly increasing, and takes on every value in the interval $[0, 1]$. Consequently, it is continuous and belongs to the class E_p. Moreover, it follows at once from the definition of $f(x)$ that $F[f(x)] = G(x)$.

Theorem 3 completes the characterization of those functions $F(x)$ which are associated with functions $f(x)$ in the class $E_p - E_p^*$.

4.3. All of the examples of functions in the class $E_p - E_p^*$ which Everett constructed give rise to correspondences with the following two properties: (1) the only sequences of integers which correspond to more than one real number are terminally periodic; and (2) the set U of those numbers which are associated uniquely with a sequence of integers has measure zero. This led him to raise the following two questions: (1) do there exist functions which in the algorithm (A) associate with more than one real number the same nonterminally periodic sequence of integers? (2) do functions exist with sets U of every measure between 0 and 1?

Stated in terms of associated functions the questions are: (1) do there exist functions whose associated functions have nonterminally periodic decimals as level values? (2) do functions exist whose associated functions have their sets of points of increase of every measure between 0 and 1? As a result of Theorem 3, we can clearly give an affirmative answer to both of these questions.

5.0. **A homeomorphism from E_p onto E_p^*.** If $f(x)$ is any function in E_p, then $F^*(x) = f(px)$ is a continuous, strictly increasing function on $[0, 1]$ to $[0, 1]$. As such it is the associated function for some function $f^*(x) \in E_p^*$. We can thus define a one-one transformation T of E_p onto E_p^* by simply setting $Tf = f^*$. If E_p and E_p^* are regarded as subsets of the metric space of continuous functions on $[0, p]$ (where the distance between two functions f and g is given by $\max |f - g|$), this transformation is, in fact, a homeomorphism as will be established in Theorems 4 and 5.

5.1. In the following sections we shall have occasion to use the theorem[3] from the theory of functions of a real variable which states that *if the functions* $f_1(x), f_2(x), \cdots$ *are continuous on the closed interval* $[a, b]$, *then a necessary and sufficient condition that* $\lim_{n\to\infty} f_n(x) = f_0(x)$ *uniformly on* $[a, b]$ *is that* $\lim_{n\to\infty}^{x\to x_0} f_n(x) = f_0(x_0)$ *for each point* x_0

[3] See, for example, Townsend, *Functions of real variables*, p. 350.

in $[a, b]$. As an easy application of this theorem we have the fact that *if* $f_1(x), f_2(x), \cdots$ *is any sequence of continuous, nondecreasing functions on* $[a, b]$ *which converges at every point of* $[a, b]$ *to a continuous function* $f_0(x)$, *then* $f_0(x)$ *is nondecreasing and the convergence is uniform on* $[a, b]$.

5.2. THEOREM 4. *If the functions* $f_n(x) \in E_p$, $n=1, 2, \cdots$, *converge on* $[0, p]$ *to the function* $f_0(x) \in E_p$, *then their corresponding associated functions* $F_n(x)$, $n=1, 2, \cdots$, *converge uniformly on* $[0, 1]$ *to* $F_0(x)$, *the associated function for* $f_0(x)$.

PROOF. It follows from §5.1 that it will be sufficient to show that $\lim_{n\to\infty} F_n(x) = F_0(x)$ pointwise on $[0, 1]$. Moreover, if we show that $\lim_{n\to\infty} F_n(x) = F_0(x)$ at each point x on the open interval $(0, 1)$ for which $0 < F_0(x) < 1$, then the convergence on the entire interval $[0, 1]$ follows from the fact that the functions $F_n(x)$ are continuous, nondecreasing with $F_n(0) = 0$ and $F_n(1) = 1$, $n = 0, 1, \cdots$.

Let γ be any point on $(0, 1)$ such that $0 < F_0(\gamma) < 1$, and let ϵ be an arbitrary positive number. Choose two numbers $A = 0.a_1a_2 \cdots a_k < F_0(\gamma)$ and $B = 0.b_1b_2 \cdots b_l > F_0(\gamma)$ such that $F_0(\gamma) - A < \epsilon$ and $B - F_0(\gamma) < \epsilon$. Then

$$f_0(a_1 + \cdots + f_0(a_k) \cdots) = \alpha < \gamma < \beta = f_0(b_1 + \cdots + f_0(b_l) \cdots).$$

§5.1 implies that $\lim_{n\to\infty} f_n(x) = f_0(x)$ uniformly on $[0, p]$, and hence, for n sufficiently large,

$$f_n(a_1 + \cdots + f_n(a_k) \cdots) = \alpha_n < \gamma < \beta_n = f_n(b_1 + \cdots + f_n(b_l) \cdots).$$

Moreover, $F_n(\alpha_n) = F_0(\alpha) = A$ and $F_n(\beta_n) = F_0(\beta) = B$. From the monotonicity of the functions $F_n(x)$ it follows that for n sufficiently large, $A \leqq F_n(\gamma) \leqq B$ and hence $|F_0(\gamma) - F_n(\gamma)| < \epsilon$.

As a corollary to Theorem 4 we have the continuity of the transformation T^{-1} from E_p^* onto E_p.

5.3. THEOREM 5. *Let* $F_0(x), F_1(x), \cdots$ *be a sequence of continuous, strictly increasing functions on* $[0, 1]$ to $[0, 1]$, *and let* $f_0(x), f_1(x), \cdots$ *be the sequence of functions in the class* E_p^* *such that* $F_n(x)$ *is the associated function for* $f_n(x)$, $n=0, 1, \cdots$. *If* $\lim_{n\to\infty} F_n(x) = F_0(x)$ *on* $[0, 1]$, *then* $\lim_{n\to\infty} f_n(x) = f_0(x)$ *uniformly on* $[0, p]$.

PROOF. Since each of the functions $F_n(x)$ is strictly increasing, $n=0, 1, \cdots$, the inverse functions $F_1^{-1}(t), F_2^{-1}(t), \cdots$ converge to $F_0^{-1}(t)$, and by §5.1 the convergence is uniform. Moreover, it will again

be sufficient to establish that $\lim_{n\to\infty} f_n(x) = f_0(x)$ at every point x on $[0, p]$. The theorem follows immediately then from §5.1 and the fact that

$$f_n(x) = F_n^{-1}\,[[x]/p + F_n(x - [x])/p],\ 0 \leqq x \leqq p,\ n = 0, 1, \cdots.$$

Theorem 5 establishes the continuity of T from E_p onto E_p^*.

THE OHIO STATE UNIVERSITY

THE NONLINEAR DIFFERENTIAL EQUATION $y'' + p(x)y + cy^{-3} = 0$

EDMUND PINNEY[1]

Among the limited number of nonlinear differential equations whose exact solutions are known is to be included

$$y''(x) + p(x)y(x) + c/y^3(x) = 0, \tag{1}$$

for c constant and $p(x)$ given. The general solution for which $y(x_0) = y_0 \neq 0$, $y'(x_0) = y_0'$ is

$$y(x) = [u^2(x) - cW^{-2}v^2(x)]^{1/2}, \tag{2}$$

where u, v form a fundamental set of solutions of the linear equation

$$y''(x) + p(x)y(x) = 0 \tag{3}$$

for which $u(x_0) = y_0$, $u'(x_0) = y_0'$, $v(x_0) = 0$, $v'(x_0) \neq 0$, where W is their Wronskian: $W = uv' - u'v = \text{const.} \neq 0$, and where the radical in (2) stands for that root which at x_0 has the value y_0.

The proof is very simple and will be omitted.

UNIVERSITY OF CALIFORNIA, BERKELEY

Received by the editors July 18, 1949.

[1] Done in connection with the Office of Naval Research contract N6 ONR 251 T.O. 2 at Stanford University. I am indebted to Prof. R. E. Langer for a simplification of the result.

THE GREEN'S FUNCTIONS FOR THE RECTANGLE OBTAINED BY THE FINITE FOURIER TRANSFORMATIONS

A. W. JACOBSON

1. Introduction. When the finite Fourier transformation method is employed, the solutions of very general boundary value problems (in contrast to initial value and mixed problems) can be expressed in terms of special functions.[1] In this paper we propose to derive, in terms of a special function to be introduced, a new formula for the Green's function for the Laplace equation $\Delta u = 0$ in a rectangle such that u vanishes on the boundary.

2. The function $B_{12}(x, u)$. In this section we introduce a key function and obtain its inverse Fourier cosine transform. If $|r| < 1$, then $\log(1+re^{i\theta}) = -\sum_{\nu=1}^{\infty}(-r)^\nu e^{i\nu\theta}/\nu$. Also if $R^2 = (1+r\cos\theta)^2 + r^2\sin^2\theta$ and $\tan\Phi = r\sin\theta/(1+r\cos\theta)$, then

$$\log Re^{i\Phi} = \log(1+re^{i\theta}) = -\sum_{\nu=1}^{\infty}\frac{(-r)^\nu e^{i\nu\theta}}{\nu}. \tag{1}$$

Equating the real part of (1), we get

$$\log R = \log[1+2r\cos\theta+r^2]^{1/2} = -\sum_{\nu=1}^{\infty}\frac{(-1)^\nu r^\nu\cos\nu\theta}{\nu}. \tag{2}$$

When $|r| \leqq 1$, the series (2) is a Fourier cosine series. The Fourier cosine transforms, defined by the operation $C\{F(x)\} = \int_0^\pi F(x)\cos nx\,dx = f_c(n)$ $(n = 0, 1, 2, \cdots)$, are except for the factor $\pi/2$ the coefficients of a Fourier cosine series. Hence, in view of (2), we get

$$C\left\{\log\frac{1}{1+2r\cos\theta+r^2}\right\} = \frac{\pi(-1)^n r^n}{n} \qquad (n = 1, 2, \cdots)$$

$$= 0 \qquad (n = 0).$$

It can be shown that $(-1)^n f_c(n) = C\{F(\pi - x)\}$. Using this fact and

Received by the editors April 18, 1949 and, in revised form, June 27, 1949.

[1] A. W. Jacobson, *A generalized convolution for the finite Fourier transformations*, Thesis (Microfilmed), University of Michigan, 1948. Also see *Solution to steady state temperature problems with the aid of a generalized fourier convolution*, Quarterly of Applied Mathematics, vol. 7 (1949).

putting $r=e^{-z}$, $\theta=x$, we obtain when $z\geqq 0$ the following formula:

$$C\left\{\log\frac{e^{z}}{2(\cosh z-\cos x)}\right\}=\frac{\pi e^{-nz}}{n}\quad(n=1,2,\cdots)$$
$$=0\qquad(n=0). \tag{3}$$

Let $b_1(n, u)$ denote the function

$$\frac{\cosh nu}{n\sinh ny_0}=\sum_{r=0}^{\infty}\frac{1}{n}\left\{e^{-[(2r+1)y_0-u]n}+e^{-[(2r+1)y_0+u]n}\right\}. \tag{4}$$

Also let $B_{12}(x, u)$ denote the inverse cosine transform $C^{-1}\{b_1(n, u)\}$, of the function $b_1(n, u)$. When $\pi-u\geqq 0$, we have according to formula (3)

$$B_{12}(x,u)=\frac{1}{\pi}\log\prod_{r=0}^{\infty}\frac{e^{2(2r+1)y_0}}{4[\cosh(2ry_0+y_0-u)-\cos x][\cosh(2ry_0+y_0+u)-\cos x]}. \tag{5}$$

3. **A boundary value problem.** Let it be required to determine the temperature $V(x, y)$ in a rectangle R when the source function $H(x, y)$ is a prescribed, sectionally continuous function of x and y and such that V vanishes on the boundary. That is, let $V(x, y)$ be the solution of the following problem:

$$V_{xx}+V_{yy}=H(x,y)\quad\text{in}\quad R,$$
$$V(x,+0)=V(x,y_0-0)=0\qquad(0<x<\pi),$$
$$V(+0,y)=V(\pi-0,y)=0\qquad(0<y<y_0). \tag{A}$$

The Fourier sine transformation, defined by the operation $S\{F(x)\}=\int_0^{\pi}F(x)\sin nx\,dx=f_s(n)$ $(n=1, 2, \cdots)$, in particular, yields when applied to the second derivative of $F(x)$ the formula

$$S\{F''(x)\}=-n^2f_s(n)+n[F(0)-(-1)^nF(\pi)].$$

Applying the sine transformation with respect to x to the equations in (A), we obtain in view of the last condition the following transformed problem:

$$\frac{d^2v}{dy^2}-n^2v(n,y)=h(n,y),$$
$$v(n,0)=v(n,y_0)=0, \tag{A'}$$

where

$$h(n,y)=S\{H(x,y)\}.$$

The solution of the transformed problem, in terms of the Green's function $g(n, y, \mu)$, is[2]

$$v(n, y) = \int_0^{y_0} g(n, y, \mu) h(n, \mu) d\mu, \tag{6}$$

where

$$g(n, y, \mu) = -\frac{\sinh n\mu \sinh n(y_0 - y)}{n \sinh ny_0} \qquad (\mu \leqq y)$$

$$= -\frac{\sinh ny \sinh n(y_0 - \mu)}{n \sinh ny_0} \qquad (\mu \geqq y)$$

or

$$g(n, y, \mu) = \frac{\cosh n(y_0 - y - \mu)}{2n \sinh ny_0} - \frac{\cosh n(y_0 - y + \mu)}{2n \sinh ny_0} \qquad (\mu \leqq y),$$

and the same expression with y and μ interchanged when $\mu \geqq y$. According to formula (4) this can be written

$$g(n, y, \mu) = 2^{-1}b_1(n, y_0 - y - \mu) - 2^{-1}b_1(n, y_0 - y + \mu), \qquad (\mu \leqq y).$$

The inverse cosine transform of $b_1(n, u)$ is the function $B_{12}(x, u)$. Hence, if $C^{-1}\{g(n, y, \mu)\} = G(x, y, \mu)$, we have

$$G(x, y, \mu) = 2^{-1}B_{12}(x, y_0 - y - \mu) - 2^{-1}B_{12}(x, y_0 - y + \mu). \tag{7}$$

The solution (6) of the transformed problem can then be written

$$S\{V(x, y)\} = \int_0^{y_0} C\{G(x, y, \mu)\} S\{H(x, \mu)\} d\mu. \tag{8}$$

The product of the two transforms in (8) can be expressed in terms of the sine transform of the convolution of the two functions. For if $P(x)$ is even and $Q(x)$ is an odd sectionally continuous function, and if $Q(x+2\pi) = Q(x)$, then

$$C\{P(x)\}S\{Q(x)\} = 2^{-1}S\{P(x) * Q(x)\},$$

where the convolution is defined thus:[3]

$$P(x) * Q(x) = \int_{-\pi}^{\pi} P(x - \lambda)Q(\lambda)d\lambda.$$

Hence the solution (8) becomes, upon inverse transformation,

[2] E. L. Ince, *Ordinary differential equations*, Dover, 1944, p. 258.

[3] R. V. Churchill, *Modern operational mathematics in engineering*, McGraw-Hill, 1944, p. 274.

$$V(x, y) = \frac{1}{2}\int_0^{y_0} G(x, y, \mu) * H(x, \mu)d\mu,$$

or

$$V(x, y) = \frac{1}{2}\int_0^{y_0}\int_{-\pi}^{\pi} G(x - \lambda, y, \mu)H(\lambda, \mu)d\lambda d\mu.$$

Since $H(\lambda, \mu)$ is extended as an odd function of λ, we get

$$V(x, y) = \frac{1}{2}\int_0^{y_0}\int_0^{\pi} [G(x - \lambda, y, \mu) - G(x + \lambda, y, \mu)]H(\lambda, \mu)d\lambda d\mu.$$

Let $\Gamma(x, y, \lambda, \mu) = 2^{-1}[G(x-\lambda, y, \mu) - G(x+\lambda, y, \mu)]$. Then the function Γ is the Green's function for the boundary value problem (A). In terms of our function $B_{12}(x, u)$, formula (7), it can be written

$$\Gamma(x, y, \lambda, \mu) = 4^{-1}[B_{12}(x - \lambda, y_0 - y - \mu) - B_{12}(x - \lambda, y_0 - y + \mu) - B_{12}(x + \lambda, y_0 - y - \mu) + B_{12}(x + \lambda, y_0 - y + \mu)]. \tag{9}$$

4. Relation between the function B_{12} and the Weierstrass's sigma function. The Green's function K for $\Delta u = 0$ in the rectangle R: $0 < x < \pi$, $0 < y < y_0$, with $u = 0$ on the boundary is in terms of the Weierstrass's sigma function σ:

$$K(x, y, \lambda, \mu) = \frac{1}{2\pi}\Re\left(\log\frac{\sigma(z - \zeta, \omega_1, \omega_2)\sigma(z + \zeta, \omega_1, \omega_2)}{\sigma(z - \bar{\zeta}, \omega_1, \omega_2)\sigma(z + \bar{\zeta}, \omega_1, \omega_2)}\right), \tag{10}$$

where

$$z = x + iy, \qquad \zeta = \lambda + i\mu, \qquad \bar{\zeta} = \lambda - i\mu, \qquad \omega_1 = \pi, \qquad \omega_2 = iy.^4$$

The expression for the Green's function Γ for the same problem, obtained in the preceding section, is given in terms of the function B_{12} by formula (9). The two functions, K and Γ, are equivalent. Comparing the two expressions above, it can be shown[5] that the function B_{12} is related to the sigma function. For the various arguments of B_{12} the following relations hold:

$$B_{12}(x - \lambda, y_0 - y - \mu) = -\frac{2}{\pi}\log|\sigma(z - \bar{\zeta})| + \frac{\eta_1}{\pi}[(x - \lambda)^2 - (y + \mu)^2] + C,$$

[4] R. Courant and D. Hilbert, *Methoden der mathematischen Physik* I, Berlin, 1931, p. 335.

[5] A. W. Jacobson, *A generalized convolution for the finite Fourier transformations*, Thesis, University of Michigan, 1948, pp. 76–84.

$$B_{12}(x-\lambda,\, y_0-y+\mu) = -\frac{2}{\pi}\log\,|\,\sigma(z+\bar{\zeta})\,| + \frac{\eta_1}{\pi}\left[(x-\lambda)^2-(y-\mu)^2\right]+C,$$

$$B_{12}(x+\lambda,\, y_0-y-\mu) = -\frac{2}{\pi}\log\,|\,\sigma(z+\zeta)\,| + \frac{\eta_1}{\pi}\left[(x+\lambda)^2-(y+\mu)^2\right]+C,$$

and

$$B_{12}(x+\lambda,\, y_0-y+\mu) = -\frac{2}{\pi}\log\,|\,\sigma(z-\zeta)\,| + \frac{\eta_1}{\pi}\left[(x+\lambda)^2-(y-\mu)^2\right]+C,$$

where the constants η_1 and C are defined as follows:

$$\eta_1 = \frac{\pi}{12} - \frac{\pi}{2}\sum_1^\infty \frac{1}{\sinh^2 \nu y_0},$$

and

$$C = \frac{1}{\pi}\log 2 - \frac{4}{\pi}\log\prod_{\nu=1}^{\infty}(1-e^{-2\nu y_0}).$$

WAYNE UNIVERSITY

ON THE TWO-DIMENSIONAL DERIVATIVE OF A COMPLEX FUNCTION

VINCENT C. POOR

1. Introduction. We are considering here the two-dimensional derivative of the complex function $f(z)=u(x, y)+iv(x, y)$ over a domain D of the complex plane. This special case of higher-dimensional derivatives was treated by Cioranescu [1][1] and by making the limit independent of direction he arrived at a unique derivative, and in this sense monogenic. The class of functions arising was studied recently in the Bulletin by Haskell [2] and by Reade [3], in which some interesting results were obtained using certain mean value theorems.

The purpose of this paper is briefly to analyze and synthesize these results using a new and different viewpoint, and to characterize more explicitly this class of polygenic functions. In particular in extending these results a second class of monogenic functions will be introduced. The whole setup leads to a complete dual or analogue to the first class of functions. This dual situation will be developed; sufficient details to exhibit this completeness will be given.

AREOLAR MONOGENIC FUNCTIONS

2. Areolar monogeneity. Cioranescu arrives at this two-dimensional derivative by selecting a rectangle in D with vertices z, z_1, z', z_2 in positive order with ϕ as slope angle of one side. He then considers the

$$\lim_{z_1, z_2 \to z} \frac{f(z') - f(z_1) - [f(z_2) - f(z)]}{(z_1 - z)(z_2 - z)}. \tag{2.1}$$

If h and k are the lengths of the sides of the rectangle, the numerator, a function of h and k, may be expanded into a double power series in h and k. The denominator will contain the area hk and an exponential factor function of ϕ. The limit is then taken and those terms involving the direction ϕ are set equal to zero, making the limit or two-dimensional derivative unique and independent of ϕ. Complex functions possessing this derivative at every point of the domain D have been called "areolar monogenic" in the domain. This type of monogeneity is characterized by the equations

Presented to the Society, April 17, 1949; received by the editors July 9, 1949 and, in revised form, August 10, 1949.

[1] Numbers in brackets refer to the bibliography at the end of the paper.

$$2u_{xy} = -(v_{xx} - v_{yy}), \qquad 2v_{xy} = u_{xx} - u_{yy} \tag{2.2}$$

or according to Cioranescu these functions are solutions of the differential equation

$$\frac{\partial^2 f}{\partial \alpha^2} = 0 \tag{2.3}$$

which is the equivalent of (2.2). The derivative itself is $v_{xy} - iu_{xy}$ while u and v are biharmonic functions. The notation $\partial f/\partial \alpha$ is used for the Pompeiu [4] areal (or areolar) derivative.

3. **Basic theorems.** We turn now to some fundamental theorems, which will be useful in the sequel.

THEOREM 1. *If $f(z)$ possesses a differential of the form $df = f_x dx + f_y dy$ in a domain D of the complex plane, then for the circulation $\int_c f(z)dz$ around every rectifiable closed curve c in D to exist and be finite it is necessary and sufficient that the areal derivative $\partial f/\partial \alpha$ exist and be finite at every point of D.*

The proof given in another paper will be omitted here; however, the proof leads to the circulation theorem.

$$\frac{1}{2\pi i}\int_c f(z)dz = \frac{1}{\pi}\int_\sigma \frac{\partial f}{\partial \alpha}\, d\sigma \tag{3.1}$$

as a consequence.

COROLLARY. *If $f(z)$ is continuous, the necessary and sufficient condition that $f(z)$ be analytic is that $\partial f/\partial \alpha = 0$.*

The vanishing of the areal derivative is identical with the Cauchy-Riemann equations.

THEOREM 2. *If $f(z)$ is continuous in D, then the necessary and sufficient condition for $f(z)$ to be areolar monogenic in D is that the areal derivative $\partial f/\partial \alpha$ be analytic in D.*

The necessity. Here $f(z)$ is areolar monogenic by hypothesis; we wish to prove that $\partial f/\partial \alpha$ is analytic.

We construct a function $f(z, z')$ of two independent variables such that when z' is replaced by $\bar{z}$, $f(z, z')$ becomes $f(z)$. We first observe that $\partial f/\partial \alpha = \partial f(z, z')/\partial z'\big|_{z'=\bar{z}}$. We now solve the defining equation

$$\frac{\partial^2 f}{\partial \alpha^2} = \frac{\partial^2 f(z, z')}{\partial z'^2}\bigg|_{z'=\bar{z}} = 0$$

for $f(z)$; or

$$\frac{\partial^2 f(z, z')}{\partial z'^2} = 0$$

so that $\partial f/\partial z' = g(z)$ and

$$f(z, z') = z'g(z) + h(z)$$

where $g(z)$ and $h(z)$ are arbitrary functions of z subject to the defining equation $\partial^2 f(z, z')/\partial z'^2 = 0$. Hence

$$f(z) = \bar{z}g(z) + h(z). \tag{3.2}$$

Evidently then

$$\frac{\partial f(z)}{\partial \alpha} = g(z), \qquad \frac{\partial h}{\partial \alpha} = 0; \qquad \frac{\partial g}{\partial \alpha} = 0,$$

and since f is continuous $g(z)$ and $h(z)$ are analytic in D and this makes the areal derivative $\partial f/\partial \alpha$ analytic in D, which was to be proved.

The sufficiency. By hypothesis $\partial f/\partial \alpha = g(z)$, an analytic function of z in D, or $\partial f(z, z')/\partial z'\big|_{z'=z} = g(z) = \partial f/\partial z'$. Hence

$$\frac{\partial^2 f(z, z')}{\partial z'^2} = 0 = \frac{\partial^2 f(z, z')}{\partial z'^2}\bigg|_{z'=z} = \frac{\partial^2 f}{\partial \alpha^2} = 0,$$

or $f(z)$ is areolar monogenic in D.

A simple alternate proof is to apply the corollary under Theorem 1; since $g(z)$ is analytic $\partial g(z)/\partial \alpha = 0 = \partial^2 f/\partial \alpha^2$.

THEOREM 3. *If $f(z)$ is continuous in D, then for $f(z)$ to be areolar monogenic in D it is necessary and sufficient that*

$$f(z) = \bar{z}g(z) + h(z) \tag{3.3}$$

where g and h are analytic functions of z.

The necessity. This part of Theorem 3 was established (3.2) in the proof of Theorem 2.

The sufficiency. Here (3.3) is given or

$$f(z, z') = z'g(z) + h(z)$$

while

$$\frac{\partial f}{\partial \alpha} = \frac{\partial f(z, z')}{\partial z'}\bigg|_{z'=z} = g(z),$$

then it is evident that $\partial^2 f(z)/\partial\alpha^2 = 0$, or $f(z)$ is areolar monogenic.

4. **Related theorems.** We wish to show here how some of the results obtained by Haskell [2] and by Reade [3] are related to the theorems obtained above. The principal theorem of the Haskell paper, though differing in appearance, is comprehended in Theorem 2 above. This is also true of Haskell's Theorem 2 for the domain $D-E$, where E is a countable set. This is then extended to D through the Menchoff [5] theorem.

The consideration of other theorems requires the application of the following

LEMMA. *If $\sigma(z, r)$ is a circular disk, center z, radius r, in D, then the average or mean value of the analytic function $f(z)$ integrated over the area of the disk is equal to the value of the function at the center of the disk and is given by the equation*

$$f(z) = \frac{1}{\pi r^2}\int_{\sigma(z,r)} f(\zeta)d\sigma. \tag{4.1}$$

PROOF. Beckenbach and Reade [8] have observed that if $f(z)$ is a harmonic polynomial, equation (4.1) or the lemma is valid. To extend this result to analytic functions, we call $f_n(z)$ the sum of the first n terms of the series representation of the analytic function; this gives

$$\frac{1}{\pi r^2}\int_{\sigma(z,r)} f_n(\zeta)d\sigma = f_n(z).$$

But this relation constitutes two identical sequences of harmonic polynomials; their limits $n\to\infty$ are therefore equal. This proves the lemma.

Briefly, we state that the lemma is applicable when $\sigma(z, r)$ is replaced by the area $P(z, r)$ of a regular n-gon which circumscribes the circle [8].

Continuing then: Theorem A and Theorem 1 of the Reade paper [2] require that there exist a function

$$g(z) = \frac{1}{2\pi i r^2}\int_{c(z,r)} f(\zeta)d\zeta \tag{4.2}$$

analytic in D as the necessary and sufficient condition for $f(z)$ to be areolar monogenic. The notation $c(r, z)$ is used for the circle radius r, center at z.

In the light of Theorem 2 we need only show that $g(z)=\partial f/\partial\alpha$. We first transform $g(z)$ by the circulation theorem (3.1) so that

$$(4.3) \qquad g(z) = \frac{1}{2\pi ir^2}\int_{c(z,r)} f(\zeta)d\zeta = \frac{1}{\pi r^2}\int_{\sigma(z,r)} \frac{\partial f}{\partial \alpha}\, d\sigma,$$

σ here being the circular area. For $f(z)$ areolar monogenic, $\partial f/\partial\alpha$ is analytic by Theorem 2 above. The lemma just proved makes the right member of (4.3) equal to $\partial f/\partial\alpha$ at z, so that $g(z)$ is analytic. Conversely, if $g(z)$ is analytic, the limit as $r\to 0$ of the right member of (4.2) is by definition $\partial f/\partial\alpha$. This makes $\partial f/\partial\alpha$ analytic and $f(z)$ areolar monogenic.

The argument just concluded is also a proof of the Haskell analogue to the Morera theorem (Theorem 3). Similarly, Reade's Theorem 2 is obtainable by making some adjustments. In (4.3) we replace πr^2 by P, the area of a regular n-gon, and $c(r, z)$ by $p(r, z)$, the perimeter of the n-gon where r is the radius of the inscribed circle with z as center. The extension of the above lemma makes the right member of the adjusted equation (4.3) equal to $\partial f/\partial\alpha$ at z, while the limit as $r\to 0$ of the adjusted (4.2) is the areal derivative again by definition. This theorem is essentially a repetition of Reade's Thoerem 1.

Mean monogenic functions

5. **Mean monogeneity.** The class of mean monogenic functions may be arrived at in various ways. The following direct derivation seems the most satisfactory. Briefly then we modify definition (2.1) by replacing the denominator by its conjugate in which case the new limit becomes

$$(5.1) \qquad \begin{aligned} &[f_{xy}\cos 2\phi - 2^{-1}(f_{xx} - f_{yy})\sin 2\phi]e^{i(2\phi+\pi/2)} \\ &\quad = u_{xx} - u_{yy} - 2v_{xy} + i(2u_{xy} + v_{xx} - v_{yy}) \\ &\qquad + [2u_{xy} - v_{xx} + v_{yy} + i(2v_{xy} + u_{xx} - u_{yy})]e^{i(4\phi+\pi/2)}, \end{aligned}$$

omitting the many details.

To make this limit independent of direction we set the coefficient of the exponential function equal to zero making the limit independent of direction with the result that

$$(5.2) \qquad 2u_{xy} = v_{xx} - v_{yy}, \qquad 2v_{xy} = -(u_{xx} - u_{yy})$$

or its equivalent

$$(5.3) \qquad \partial^2 f(z)/\partial\beta^2 = 0$$

where $\partial f/\partial\beta$ is the Kasner [7] mean derivative or

$$\frac{\partial f}{\partial \beta} = \lim_{\Delta\sigma_j \to 0} \frac{-1}{2i\Delta\sigma_j}\int_{\sigma_j} f(z)d\bar{z}.$$

Solutions of (5.3) will be called *mean monogenic functions* because of this close relation with the Kasner mean derivative. The unique mean monogenic derivative itself is $-v_{zy}+iu_{zy}$ while, again, u and v are biharmonic functions.

6. **Theorems of mean monogenic functions.** We can now prove the analogue to Theorem 1. This theorem leads to the circulation theorem

$$\frac{-1}{2\pi i}\int_c f(z)d\bar{z} = \frac{1}{\pi}\int_\sigma \frac{\partial f}{\partial \beta}\,d\sigma \tag{6.1}$$

where c in D is the boundary of the area σ as before. We now state two theorems to emphasize the duality between the two classes of functions considered here.

THEOREM 4. *If $f(z)$ is continuous in D, then for $f(z)$ to be mean monogenic in D it is necessary and sufficient that the mean derivative $\partial f(z)/\partial\beta$ be an analytic function of $\bar{z}$ in D.*

THEOREM 5. *If $f(z)$ is continuous in D, then the necessary and sufficient condition for $f(z)$ to be mean monogenic in D is that it have the form given by the equation*

$$f(z) = zp(\bar{z}) + q(\bar{z}) \tag{6.2}$$

where p and q are analytic functions of $\bar{z}$.

The proofs of these two theorems will be omitted since the arguments follow those of Theorems 2 and 3, respectively. Analogues to the theorems of Haskell and of Reade may be stated and proved for the class of mean monogenic functions.

7. **Duality.** We shall here try to expose the completeness of the duality existing between the areolar and mean monogenic functions. In addition to the two theorems just stated together with the analogues just mentioned we supplement their weight by

THEOREM 6. *If $f(z)$ is areolar monogenic and at the same time mean monogenic in D, then*

$$f(z) = az\bar{z} + bz + c\bar{z} + d. \tag{7.1}$$

In our proof we have from our hypothesis the relation

$$f(z) = \bar{z}g(z) + h(z) \equiv zp(\bar{z}) + q(\bar{z}) \tag{7.2}$$

obtained from (3.3) and (6.2).

Since the highest degree term in (7.2) is $z\bar{z}$, it follows that $g(z)$

$=a_1z+b_1$ and $p(\bar{z})=a_2\bar{z}+b_2$; (7.2) then becomes

$$az\bar{z} + b_1\bar{z} + h(z) \equiv az\bar{z} + b_2z + q(\bar{z})$$

if we write a for a_1 and a_2. From this we have $h(z)=c_1z+d$ and $q(\bar{z})=c_2\bar{z}+d$; the adjustment of the constants gives (7.1).

To strengthen our conclusion we may note that if $f(z)$ is in the general form for an areolar monogenic function given by (3.3), then

$$\bar{f}(z) = z\bar{g}(z) + \bar{h}(z) = zp(\bar{z}) + q(\bar{z}),$$

or if the coefficients of g and h in expanded form are real numbers, then $\bar{f}=zg(\bar{z})+h(\bar{z})$. In any case $\bar{f}(z)$ is the standard form for mean monogenic functions. On the other hand when $f(z)$ is given by (6.2)

$$\bar{f}(z) = \bar{z}\bar{p}(\bar{z}) + \bar{q}(\bar{z}) = \bar{z}g(z) + h(z)$$

or equals $\bar{z}p(z)+q(z)$ for real coefficients as before. Here $\bar{f}$ is areolar monogenic. In other words, every function of one class has an exact correspondent in the other. We are thus justified in saying that to every theorem pertinent to one class of functions there is an exact dual theorem valid for the other class.

With the above analysis we may conclude that the duality between areolar and mean monogenic functions is complete. The forms given by (3.3) and (6.2) also completely characterized these two classes of functions.

Bibliography

1. N. Cioranescu, *Sur la définition de la mongénéité et les fonctions monogènes areolariment*, Mathematica vol. 12 (1935) pp. 26–30.

2. R. N. Haskell, *Areolar monogenic functions*, Bull. Amer. Math. Soc. vol. 52 (1946) pp. 332–337.

3. M. O. Reade, *On areolar monogenic functions*, Bull. Amer. Math. Soc. vol. 53 (1947) pp. 98–103.

4. D. Pompeiu, *Sur une classe de fonctions d'une variable complex*, Rend. Circ. Math. Palermo vol. 33 (1912) pp. 108–113.

5. D. Menchoff, *Les conditions de monogénéité*, Actualités Scientifiques et Industrielle, no. 329, Paris, 1936.

6. Vincent C. Poor, *Complex functions possessing differentials*, Amer. J. Math. vol. 68 (1946) pp. 147–160.

7. E. Kasner, *General theory of polygenic or non-monogenic functions. The derivative congruence of circles*, Proc. Nat. Acad. Sci. U.S.A. vol. 14 (1928) pp. 75–83.

8. E. F. Beckenbach and M. O. Reade, *Mean values and harmonic polynomials*, Trans. Amer. Math. Soc. vol. 53 (1943) pp. 230–238.

University of Michigan

NOTE ON PRESERVATION OF MEASURABILITY

ROY B. LEIPNIK

If m measures X (that is, m is a countably subadditive function on subsets of X to the non-negative reals), and if f is a function on subsets of Y to subsets of X, it is natural to inquire into the behavior of the composite function n, where $n(A)=m(f(A))$ for each $A\subset Y$. Rather obvious sufficient conditions are given below for n to measure Y, and for a subset A of Y to be n-measurable.

Several definitions are necessary. A subset C of X is called m-measurable-F if and only if $m(T\cap D)=m(T\cap D\cap C)+m(T\cap D\cap(X-C))$ whenever $T\subset X$ and $D\in F$. A subset B of Y is called a set of m-continuity-f if and only if $m(f(B)\cap f(Y-B))=0$. Suppose henceforth that (*) $m(f(A)\cap(X-\bigcup_{B\in\mathfrak{C}}f(B)))=0$ whenever $A\subset Y$ and $\mathfrak{C}$ is a countable covering of A by subsets of Y.

THEOREM 1. *If* (*), *then* n *measures* Y.

PROOF. If $A\subset Y$, and $\mathfrak{C}$ is a countable covering of A by subsets of Y, then $n(A)=m(f(A))\leqq m(f(A)\cap\bigcup_{B\in\mathfrak{C}}f(B))+m(f(A)\cap(X-\bigcup_{B\in\mathfrak{C}}f(B)))=m(\bigcup_{B\in\mathfrak{C}}(f(A)\cap f(B)))\leqq\sum_{B\in\mathfrak{C}}m(f(A)\cap f(B))\leqq\sum_{B\in\mathfrak{C}}m(f(B))=\sum_{B\in\mathfrak{C}}n(B)$, so that n measures Y.

LEMMA 1. *If* (*), $B\subset Y$, *and* $C\subset Y$, *then* $n(B\cap C)\leqq m(f(B)\cap f(C))$.

PROOF. Since $B\cap C\subset B$ and $B\cap C\subset C$, $m(f(B\cap C)\cap(X-f(B)\cap f(C)))=m(f(B\cap C)\cap(X-f(B)))\cup(f(B\cap C)\cap(X-f(C)))\leqq m(f(B\cap C)\cap(X-f(B)))+m(f(B\cap C)\cap(X-f(C)))=0$. Hence $n(B\cap C)\leqq m(f(B\cap C)\cap(f(B)\cap f(C)))+m(f(B\cap C)\cap(X-f(B)\cap f(C)))\leqq m(f(B)\cap f(C))$.

THEOREM 2. *If* (*), A *is a set of* m*-continuity-*f, *and* $f(A)$ *is* m*-measurable-range* f, *then* A *is* n*-measurable.*

PROOF. Suppose $T\subset Y$. By Lemma 1, $n(T\cap A)\leqq m(f(T)\cap f(A))$ and $n(T\cap(Y-A))\leqq m(f(T)\cap f(Y-A))\leqq m(f(T)\cap f(Y-A)\cap f(A))+m(f(T)\cap f(Y-A)\cap(X-f(A)))\leqq m(f(A)\cap f(Y-A))+m(f(T)\cap(X-f(A)))$. By hypothesis, $m(f(A)\cap f(Y-A))=0$ and $m(f(T)\cap f(A))+m(f(T)\cap(X-f(A)))=m(f(T))=n(T)$. Therefore $n(T\cap A)+n(T\cap(Y-A))\leqq n(T)$.

REFERENCE

1. T. J. McMinn, *Restricted measurability*, Bull. Amer. Soc. vol. 54 (1948) p. 1106.

INSTITUTE FOR ADVANCED STUDY

Received by the editors August 5, 1949.

A NOTE ON INTERPOLATION

GEORGE KLEIN

Let $f(x)$ be a function of period 2π and let $I_n(x; f)$ be the trigonometric polynomial of order n coinciding with f at $2n+1$ equidistant points

$$\{x_0, x_1, \cdots, x_{2n}\} = \{x_0, x_0 + h, \cdots, x_0 + 2nh\},$$

$$h = h_n = \frac{2\pi}{2n+1}, \tag{1}$$

of the interval $[0, 2\pi]$. The familiar formula of Lagrange for $I_n(x; f)$ can be written in the form

$$I_n(x; f) = \frac{1}{\pi}\int_0^{2\pi} f(t) D_n(x - t)\, d\omega_{2n+1}(t) \tag{2}$$

where $D_n(u) = 2^{-1}\sum_{\nu=-n}^{n} e^{i\nu u}$ is the Dirichlet kernel and $\omega_{2n+1}(t)$ is a step function constant in the interior of the intervals $[x_0+kh, x_0+(k+1)h]$ and having a jump h at the points x_0+kh.[1] Thus $\omega_{2n+1}(t)$ is determined up to an arbitrary additive constant. It is well known that $I_n(x; f)$ need not tend to $f(x)$ even if f is everywhere continuous.

S. Bernstein[2] pointed out that the situation is considerably improved if we consider the trigonometric polynomial of order n assuming at the points x_k the values

$$2^{-1}\{f(x_k - h/2) + f(x_k + h/2)\} \qquad (k = 0, 1, \cdots, 2n). \tag{3}$$

In this case, for every bounded f, the interpolating polynomials are contained between the upper and lower bounds of f and converge to f at every point of continuity, the convergence being uniform over every closed interval of continuity. The same holds if we take the polynomials assuming at the points x_k the values

$$2^{-2}\{f(x_k - h) + 2f(x_k) + f(x_k + h)\} \tag{4}$$

or

$$2^{-3}\{f(x_k - 3h/2) + 3f(x_k - h/2) + 3f(x_k + h/2) + f(x_k + 3h/2)\} \tag{5}$$

Received by the editors June 21, 1949.

[1] This notation seems to have first been used by J. Marcinkiewicz, *On interpolation polynomials* (in Polish), Wiadomosci Matematyczne vol. 39 (1935) pp. 207–221.

[2] S. Bernstein, *Sur une modification de la formule d'interpolation de Lagrange*, Communications of the Mathematical Society of Kharkov vol. 5 pp. 49–57.

$(k=0, 1, \cdots, 2n)$, and so on.

It is natural to expect that in the cases (4), (5), and so on, the convergence of the interpolating polynomials to f is stronger than is (3), and we shall show that this is so by proving that then even the differentiated polynomials converge to the derivatives of f, provided the latter exist (a fact which has practical uses). More precisely, our result is as follows:

THEOREM. *Let p be a positive integer and let f be a bounded measurable function of period 2π which has at the point x' a derivative of order $j<p$. Let $T_{n,p}(x; f)$ be the trigonometric polynomial of order n which assumes at the points $x_k=x_0+kh$ $(k=0, 1, \cdots, 2n; h=h_n=2\pi/(2n+1))$ the values*

$$(6)\quad f_k=\frac{1}{2^p}\sum_{i=0}^{p} C_{p,i} f\left(x_k+\left(i-\frac{p-1}{2}\right)h\right), \quad \text{where } C_{p,i}=\frac{p!}{i!\,(p-i)!}$$

Then

$$(7)\qquad \lim_{n\to\infty}\frac{d^j}{dx^j}T_{n,p}(x'; f)=f^{(j)}(x')\,.$$

In this theorem the number $x_0=x_0^{(n)}$ may depend on n. The derivative here is taken in the generalized sense due to Peano and de la Vallee-Poussin, that is, if for small values of $|t|$,

$$f(x+t)=a_0+\frac{a_1}{1!}t+\frac{a_2}{2!}t^2+\cdots+\frac{a_{j-1}}{(j-1)!}t^{j-1}+\frac{(a_j+\epsilon_t)}{j!}t^j$$

where the a's are constants and $\epsilon_t\to 0$ as $t\to 0$, then f is said to have a jth generalized derivative at x, which is equal to a_j.

Using (2) one easily sees that

$$\begin{aligned}T_{n,p}(x; f)&=\frac{1}{2^p}\sum_{j=0}^{p} C_{p,j}\frac{1}{\pi}\int_0^{2\pi} f\left(t+\left(j-\frac{p-1}{2}\right)h\right)D_n(x-t)\,d\omega_{2n+1}(t)\\&=\frac{1}{2^p}\sum_{j=0}^{p} C_{p,j}\frac{1}{\pi}\int_0^{2\pi} f(t)D_n\left(x+\left(j-\frac{p-1}{2}\right)h-t\right)\\&\quad\cdot d\omega_{2n+1}\left(t-\left(j-\frac{p-1}{2}\right)h\right).\end{aligned}$$

If p is odd, then $d\omega_{2n+1}(t-(j-(p-1)/2)h)$ may be replaced by $d\omega_{2n+1}(t)$, and if p is even,—by $d\omega_{2n+1}(t+h/2)$. Setting, therefore,

$$\omega^*_{2n+1}(t) = \omega_{2n+1}(t) \qquad \text{if} \quad p \text{ is odd,}$$

and

$$\omega^*_{2n+1}(t) = \omega_{2n+1}(t + h/2) \quad \text{if} \quad p \text{ is even,}$$

we see that

$$T_{n,p}(x; f) = \frac{1}{\pi}\int_0^{2\pi} f(t)D_{n,p}(x - t)d\omega^*_{2n+1}(t) \tag{8}$$

where

$$\begin{aligned} D_{n,p}(u) &= \frac{1}{2^p}\sum_{j=0}^{p} C_{p,j}D_n\left(u + \left(j - \frac{p-1}{2}\right)h\right) \\ &= \frac{1}{2^p}\sum_{j=0}^{p} C_{p,j}\,\frac{1}{2}\sum_{\nu=-n}^{n} e^{i\nu(u+(j-(p-1)/2)h)} \\ &= \frac{1}{2}\sum_{\nu=-n}^{n} e^{i\nu u}\,\frac{1}{2^p}\sum_{j=0}^{p} C_{p,j}e^{i(j-(p-1)/2)\nu h} \\ &= \frac{1}{2}\sum_{\nu=-n}^{n} e^{i\nu u}\cos^p\frac{\nu h}{2} = \frac{1}{2} + \sum_{\nu=1}^{n}\cos \nu u \cos^p\frac{\nu h}{2}. \end{aligned} \tag{9}$$

In formula (8) we can write $d\omega_{2n+1}(t)$ instead of $d\omega^*_{2n+1}(t)$, since this amounts to translating x_0 by $h/2$ for p even, and as we have already stated our x_0 is arbitrary. If the νth partial sum of the polynomial I_n (see (2)) is denoted by $I_{n,\nu}$, we have

$$I_{n,\nu}(x; f) = \frac{1}{\pi}\int_0^{2\pi} f(t)D_\nu(x - t)d\omega_{2n+1}(t). \tag{10}$$

Let us apply summation by parts to the last sum in (9). We get

$$D_{n,p}(u) = \sum_{\nu=0}^{n-1} D_\nu(u)\Delta\cos^p\frac{\nu h}{2} + D_n(u)\cos^p\frac{nh}{2},$$

and this inserted in (8) gives (see (10))

$$T_{n,p}(x; f) = \sum_{\nu=0}^{n-1} I_{n,\nu}(x; f)\Delta\cos^p\frac{\nu h}{2} + I_n(x; f)\cos^p\frac{nh}{2}. \tag{11}$$

For any sequence $a_0, a_1, \cdots$, we use the notation $\Delta a_\nu = a_\nu - a_{\nu+1}$, $\Delta^s a = \Delta(\Delta^{s-1}a_\nu)$.

Let $S^k_{n,\nu}$ and $I^k_{n,\nu}$ denote the kth Cesaro sums and the kth Cesaro means, respectively, of the sequence $I_{n,\nu}$ for n fixed. Thus

$$S^0_{n,\nu} = I_{n,\nu},\ S^k_{n,\nu} = S^{k-1}_{n,0} + S^{k-1}_{n,1} + \cdots + S^{k-1}_{n,\nu},$$

$$I^k_{n,\nu} = \frac{S^k_{n,\nu}}{A^k_\nu}, \text{ with } A^k_\nu = C_{\nu+k,\nu}.$$

Applying to the right side of (11) repeated summation by parts we easily get

$$\begin{aligned} T_{n,p}(x;f) &= \sum_{\nu=0}^{n-2} S^1_{n,\nu}(x;f)\Delta^2 \cos^p \frac{\nu h}{2} + S^1_{n,n-1}(x;f)\Delta \cos^p \frac{(n-1)h}{2} \\ &\quad + S^0_{n,n} \cos^p \frac{nh}{2} \\ &= \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &= \sum_{\nu=0}^{n-p-1} S^p_{n,\nu}(x;f)\Delta^{p+1} \cos^p \frac{\nu h}{2} \\ &\quad + \sum_{k=0}^{p} S^{p-k}_{n,n-p+k}(x;f)\Delta^{p-k} \cos^p \frac{(n-p+k)h}{2} \\ &= \sum_{\nu=0}^{n-p-1} I^p_{n,\nu}(x;f)A^p_\nu \Delta^{p+1} \cos^p \frac{\nu h}{2} \\ &\quad + \sum_{k=0}^{p} I^{p-k}_{n,n-p+k}(x;f)A^{p-k}_{n-p+k}\Delta^{p-k} \cos^p \frac{(n-p+k)h}{2}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{d^j}{dx^j} T_{n,p}(x';f) &= \sum_{\nu=0}^{n-p-1} \frac{d^j}{dx^j} I^p_{n,\nu}(x';f)A^p_\nu \Delta^{p+1} \cos^p \frac{\nu h}{2} \\ &\quad + \sum_{k=0}^{p} \frac{d^j}{dx^j} I^{p-k}_{n,n-p+k}(x';f)A^{p-k}_{n-p+k}\Delta^{p-k} \cos^p \frac{(n-p+k)h}{2}. \end{aligned} \tag{12}$$

By way of making clear our reasons for expressing $d^jT_{n,p}(x';f)/dx^j$ in the form (12), we observe first that if f happens to be a trigonometric polynomial P, then the theorem is true. For then, as seen from (6), $T_{n,p}(x;f) = (1/2^p)\sum_{j=0}^{p} C_{p,j}\, P(x+(j-(p-1)/2)h)$. Differentiating this equation j times and letting $n\to\infty$, we get $T^{(j)}_{n,p}(x';f)\to p^{(j)}(x')$. Furthermore we can find a polynomial P which has the same derivatives as f at the point x'. Subtracting P from f, we can suppose that $f(x'+t) = o(|t|^j)$ if $f^{(j)}(x')$ exists. Our task is then to show that for such f the expression (12) approaches zero as $n\to\infty$.

To this end we shall evaluate the orders of the various terms on the right side of (12) for such f. In particular we shall show that

$$(13)\qquad \frac{d^j}{dx^j} I_{n,\nu}^{p}(x'; f) \to 0 \quad \text{as } \nu \to \infty,\ \nu \leqq n, \text{ for } j = 0, 1, \cdots, p-1.$$

Once (13) is established, it will immediately follow that

$$(14)\qquad \frac{d^j}{dx^j} I_{n,\nu}^{p-k}(x'; f) = o(\nu^k),\ k = 0, 1, \cdots, p; j = 0, 1, \cdots, p-1.$$

For $k=0$, (14) is the assertion (13). We assume that (14) has been proved for $k=0, 1, \cdots, k_1$. Then

$$\frac{d^j}{dx^j} I_{n,\nu}^{p-(k_1+1)}(x'; f) = -\frac{1}{A_\nu^{p-(k_1+1)}} \Delta A_{\nu-1}^{p-k_1} \frac{d^j}{dx^j} I_{n,\nu-1}^{p-k_1}(x'; f)$$

$$= O\left(\frac{A_\nu^{p-k_1}}{A_\nu^{p-k_1-1}} \frac{d^j}{dx^j} I_{n,\nu}^{p-k_1}(x'; f)\right)$$

$$= O(\nu) \cdot o(\nu^{k_1}) = o(\nu^{k_1+1}).$$

This proves (14), given (13).

We shall also need the facts

$$(15)\qquad A_\nu^p \Delta^{p+1} \cos^p \frac{\nu h}{2} = O\left(\frac{1}{n}\right)$$

and

$$(16)\qquad A_{n-p-k}^{p-k} \Delta^{p-k} \cos^p \frac{n-p+k}{2} h = O\left(\frac{1}{n^k}\right).$$

These follow from the fact that $A_\nu^p = O(\nu^p)$ and that $\Delta^{p+1} \cos^p (\nu h/2) = O(1/n^{p+1})$, $\Delta^{p-k} \cos^p (((n-p+k)/2)h) = O(1/n^p)$, the last two being consequences of the mean value theorem and the fact that $\lim_{n\to\infty} ((n-p+k)/2)h_n = \pi/2$.

With this information we show that $d^j T_{n,p}(x'; f)/dx^j \to 0$ for f of the prescribed type, by denoting in the expression (12) the two sums by A and B, where

$$A = \sum_{\nu=0}^{n-p-1} \frac{d^j}{dx^j} I_{n,\nu}^{p}(x'; f) A_\nu^p \Delta^{p+1} \cos^p \frac{\nu h}{2}$$

$$= \sum_{\nu=0}^{n-p-1} o(1) \cdot O\left(\frac{1}{n}\right) = o(1), \qquad \text{by (13) and (15),}$$

and where

$$B = \sum_{k=0}^{p} \frac{d^j}{dx^j} I_{n,n-p+k}^{p-k}(x'; f) A_{n-p+k}^{p-k} \Delta^{p-k} \cos^p \frac{n-p+k}{2} h$$

$$= \sum_{k=0}^{p} o(n^k) \cdot O\left(\frac{1}{n^k}\right) = o(1), \qquad \text{by (14) and (16).}$$

We see therefore that our theorem will be proved if (13) is established, and we now proceed to that.

We note first that

$$I_{n,\nu}^{p}(x; f) = \frac{1}{\pi} \int_0^{2\pi} f(t) K_\nu^p(x-t) d\omega_{2n+1}(t),$$

where $K_\nu^p(u)$ is the pth Cesaro mean of order ν of the Dirichlet kernel $D_n(u)$. Thus it is clear that

$$\frac{d^j}{dx^j} I_{n,\nu}^{p}(x'; f) = \frac{1}{\pi} \int_0^{2\pi} f(t) \frac{d^j}{dx^j} K_\nu^p(x-t) d\omega_{2n+1}(t)$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x'+t) \frac{d^j}{dt^j} K_\nu^p(t) d\omega_{2n+1}(x'+t).$$

Hence we have

$$\left| \frac{d^j}{dx^j} I_{n,\nu}^{p}(x'; f) \right| \leqq \frac{1}{\pi} \int_{-\pi}^{\pi} \epsilon_t |t|^j \cdot \left| \frac{d^j}{dt^j} K_\nu^p(t) \right| d\omega_{2n+1}(x'+t), \tag{17}$$

where ϵ_t is non-negative and approaches zero with t.

It is shown in Zygmund, *Trigonometrical series*, p. 259, that

$$\left| \frac{d^j}{dt^j} K_\nu^p(t) \right| \leqq C \nu^{j+1} \tag{18}$$

and that $|d^j K_\nu^p(t)/dt^j|$ is less than the sum of the three expressions

$$C_1 \sum_{k=1}^{s} \frac{\nu^{-k}}{|t|^{j+1+k}}, \qquad C_2 \sum_{\mu=0}^{j} \frac{\nu^{\mu-p}}{|t|^{p+1+j-\mu}}, \qquad C_3 \sum_{\mu=0}^{j} \frac{\nu^{-s+\mu}}{|t|^{s+j-\mu+1}},$$

where the C's are constants, s is any integer greater than $p+j$, and $|t| \leqq \pi$. It follows easily from this fact that

$$|t|^j \cdot \left| \frac{d^j}{dt^j} K_\nu^p(t) \right| \leqq C \frac{1}{\nu t^2} \qquad \text{for } \frac{1}{\nu} \leqq |t| \leqq \pi. \tag{19}$$

We now use (18) and (19) to prove

$$(20)\qquad \frac{1}{\pi}\int_{-\pi}^{\pi} |t|^j \cdot \left|\frac{d^j}{dt^j} K_\nu^p(t)\right| d\omega_{2n+1}(x'+t) \leqq C$$

by splitting the integral in (20) into two parts, $\int_{-1/\nu}^{1/\nu} + \int_R = A + B$, where R is the complement of $[-1/\nu, 1/\nu]$ in $[-\pi, \pi]$.

$$A = \frac{1}{\pi}\int_{-1/\nu}^{1/\nu} |t|^j \left|\frac{d^j}{dt^j} K_\nu^p(t)\right| d\omega_{2n+1}(x'+t)$$

$$\leqq C\nu^{j+1}\int_{-1/\nu}^{1/\nu} |t|^j d\omega_{2n+1}(x'+t).$$

Hence

$$A \leqq C\nu^{j+1}\cdot\frac{1}{\nu^j}\int_{-1/\nu}^{1/\nu} d\omega_{2n+1}(x'+t)$$

$$= C\nu\cdot\frac{2\pi}{2n+1}\cdot\frac{2n+1}{2\pi\nu} = C.$$

Furthermore

$$B = \frac{1}{\pi}\int_R |t|^j \cdot \left|\frac{d^j}{dt^j} K_\nu^p(t)\right| d\omega_{2n+1}(x'+t),$$

so that by (19),

$$B \leqq C\int_R \frac{1}{\nu t^2} d\omega_{2n+1}(x'+t) = C\frac{1}{\nu}\sum_{x_k\in R}\frac{1}{(x'+x_k)^2}\cdot\frac{2\pi}{2n+1}.$$

Now

$$\sum_{x_k\in R}\frac{1}{(x'+x_k)^2} \leqq 2\sum_{j=0}^{\infty}\frac{1}{(1/\nu+(2\pi/(2n+1))j)^2}$$

$$= 2\left[\nu^2 + \sum_{j=1}^{\infty}\frac{1}{(1/\nu+(2\pi/(2n+1))j)^2}\right]$$

$$\leqq 2\left[\nu^2 + \frac{2n+1}{2\pi}\int_0^{\infty}\frac{du}{(1/\nu+u)^2}\right]$$

$$= 2\left[\nu^2 + \frac{2n+1}{2\pi}\nu\right].$$

Thus

$$B \leqq C\frac{1}{\nu}\cdot\frac{2\pi}{2n+1}\cdot 2\left[\nu^2 + \frac{2n+1}{2\pi}\nu\right] \leqq C',$$

and (20) is proved.

To see that the integral in (17) can be made arbitrarily small by taking ν sufficiently large, we choose a $\delta>0$ small enough so that $\epsilon_t<\epsilon$ for $|t|<\delta$, and split the integral into two parts $\int_{-\delta}^{\delta}+\int_S=A+B$, S being the complement of $[-\delta, \delta]$ in $[-\pi, \pi]$. Since by (19), $|t|^j\left|d^jK_\nu^p(t)/dt^j\right|$ approaches zero uniformly in S, we see that $B=o(1)$ as $\nu\to\infty$. Furthermore

$$A=\frac{1}{\pi}\int_{-\delta}^{\delta}\epsilon_t|t|^j\cdot\left|\frac{d^j}{dt^j}K_\nu^p(t)\right|d\omega_{2n+1}(x'+t)$$

$$\leqq\frac{\epsilon}{\pi}\int_{-\pi}^{\pi}|t|^j\left|\frac{d^j}{dt^j}K_\nu^p(t)\right|d\omega_{2n+1}(x'+t)\leqq\epsilon C, \qquad \text{by (20).}$$

Thus $A+B$ can be made arbitrarily small by choosing ν sufficiently large.

Thus (13) is established and our theorem is proved.

It is not difficult to see that if f has a jth derivative ($j<p$) at each point of an interval $a\leqq x\leqq b$ and that derivative is bounded (continuous) there, then the expression under the limit sign in (7) is uniformly bounded (convergent) in that interval.

It should be noted that a slight alteration in the proof of (20) yields the fact that if f has a pth derivative at the point x', then

$$\frac{d^p}{dx^p}T_{n,p}(x';f)=o(\log n).$$

UNIVERSITY OF CHICAGO

Books Published by the

American Mathematical Society

Colloquium Publications

35. A. C. Schaeffer and D. C. Spencer, **Coefficient regions for schlicht functions**, xvi, 311 pp. **To appear soon.** $6.00
34. J. L. Walsh, **The location of critical points of analytic and harmonic functions**, 1950, viii, 384 pp. 6.00
33. J. F. Ritt, **Differential algebra**, 1950, viii, 184 pp. 4.40
32. R. L. Wilder, **Topology of manifolds**, 1949, x, 402 pp. 7.00
31. E. Hille, **Functional analysis and semi-groups**, 1948, xii, 528 pp. 7.50
30. T. Radó, **Length and area**, 1948, vi, 572 pp. 6.75
29. A. Weil, **Foundations of algebraic geometry**, 1946, xx, 288 pp. 5.50
28. G. T. Whyburn, **Analytic topology**, 1942; reprinted, 1948, x, 280 pp. 4.75
27. S. Lefshetz, **Algebraic topology**, 1942; reprinted, 1948, vi, 393 pp. 6.00
26. N. Levinson, **Gap and density theorems**, 1940, viii, 246 pp. 4.00
25. G. Birkhoff, **Lattice theory**, 1940; enlarged and completely revised edition, 1949, xiv, 283 pp. 6.00
24. A. A. Albert, **Structure of algebras**, 1939, xii, 210 pp. 4.00
23. G. Szegö, **Orthogonal polynomials**, 1939; reprinted, 1949, x, 403 pp. 6.00
22. C. N. Moore, **Summable series and convergence factors**, 1938, vi, 105 pp. 2.00
21. J. M. Thomas, **Differential systems**, 1937, x, 118 pp. 2.00
20. J. L. Walsh, **Interpolation and approximation by rational functions in the complex domain**, 1935, x, 382 pp. 6.00
19. R. E. A. C. Paley and N. Wiener, **Fourier transforms in the complex domain**, 1934; reprinted, 1949, viii, 184 pp. +portrait plate. 3.75
18. M. Morse, **The calculus of variations in the large**, 1934; reprinted, 1948, x, 368 pp. 5.75
17. J. H. M. Wedderburn, **Lectures on matrices**, 1934; reprinted, 1946; reprinted, 1949, x, 205 pp. 3.35
16. G. A. Bliss, **Algebraic functions**, 1933; reprinted, 1948, x, 220 pp. 3.75
14. J. F. Ritt, **Differential equations from the algebraic standpoint**, 1932; reprinted, 1948, x, 172 pp. 3.00
11. D. Jackson, **The theory of approximation**, 1930; reprinted, 1946, viii, 178 pp. 3.35
10. A. B. Coble, **Algebraic geometry and theta functions**, 1929; reprinted, 1948, viii, 282 pp. 4.00
9. G. D. Birkhoff, **Dynamical systems**, 1927; reprinted, 1948, viii, 295 pp. 4.25
8. L. P. Eisenhart, **Non-Riemannian geometry**, 1927; reprinted, 1949, viii, 184 pp. 2.70
5½. O. Veblen, **Analysis situs**, 1922; second ed., 1931; reprinted, 1946, x, 194 pp. 3.35
3½. E. Kasner, **Differential-geometric aspects of dynamics**, 1913; reprinted, 1934; reprinted, 1948, iv, 117 pp. 2.50
1. H. S. White, **Linear systems of curves on algebraic surfaces**;
   F. S. Woods, **Forms of non-Euclidean space**;
   E. B. Van Vleck, **Selected topics in the theory of divergent series and of continued fractions**; 1905, xii, 187 pp. 3.00

Mathematical Surveys

4. O. F. G. Schilling, **The theory of valuations**, viii, 253 pp. $6.00
3. M. Marden, **The geometry of the zeros of a polynomial in a complex variable**, 1949, x, 181 pp. 5.00
2. N. Jacobson, **The theory of rings**, 1943; reprinted, 1949, vi, 151 pp. 3.35

Memoirs

2. J. Dieudonné, **On the automorphisms of the classical groups**, with a supplement by L. K. Hua, ii, 122 pp. **To appear soon.**
1. G. T. Whyburn, **Open mappings on locally compact spaces**, 1950, ii, 25 pp. $0.75

Proceedings of Symposia in Applied Mathematics

2. **Electromagnetic theory**, 1950, iv, 91 pp. $3.00
1. **Non-linear problems in mechanics of continua**, 1949, iv, 219 pp. 5.25

The price to members of the American Mathematical Society is 25% less than list
Volumes which are not listed above are out of print.

CONTENTS

Vol. 1, No. 5 OCTOBER, 1950 Whole No. 5

Page

A basis for free Lie rings and higher commutators in free groups. By MARSHALL HALL 575
Infinite-dimensional irreducible representations of certain groups. By F. I. MAUTNER 582
A note on the characteristic polynomials of certain matrices. By W. T. REID 584
On the matric equation $X^2+AX+XB+C=0$. By W. E. ROTH 586
Some remarks on a theorem of Erdös concerning asymptotic density. By H. N. SHAPIRO 590
Certain best possible results in the theory of Schnirelmann density. By BENJAMIN LEPSON 592
A note on isothermal nets. By V. G. GROVE 595
Deformation theory of subspaces in a Riemann space. By V. HLAVATÝ 600
Remarks on certain pathological open subsets of 3-space and their fundamental groups. By W. A. BLANKINSHIP and R. H. FOX 618
Some limiting distributions related to the sum of a random number of random variables. By EDWARD PAULSON 625
Bounds for the coefficients of univalent functions. By ARYEH DVORETZKY 629
Limiting values of subharmonic functions. By ELMER TOLSTED. 636
Generalization of a theorem of Osgood to the case of continuous approximation. By A. M. OSTROWSKI 648
Uniform approximations to a class of Bessel functions. By A. M. PEISER 650
Banach-Hausdorff limits. By W. F. EBERLEIN 662
Proof of a theorem of Jacobi. By N. J. FINE 666
A remark on the moduli of Riemann surfaces of genus 2. By P. R. GARABEDIAN 668
The representation of real numbers. By O. W. RECHARD 674
The nonlinear differential equation $y''+p(x)y+cy^{-3}=0$. By EDMUND PINNEY 681
The Green's functions for the rectangle obtained by the finite Fourier transformations. By A. W. JACOBSON 682
On the two-dimensional derivative of a complex function. By V. C. POOR 687
Note on preservation of measurability. By R. B. LEIPNIK 694
A note on interpolation. By GEORGE KLEIN 695

Articles for insertion in the Proceedings should be typewritten, double spaced, and the author should keep a complete copy. Papers in algebra and theory of numbers should be sent to NATHAN JACOBSON, Leet Oliver Memorial Hall, Yale University, New Haven 11, Conn.; in analysis to A. C. SCHAEFFER, North Hall, University of Wisconsin, Madison 6, Wis.; in other fields to G. A. HEDLUND, Leet Oliver Memorial Hall, Yale University, New Haven 11, Conn. All other communications to the editors should be addressed to the Managing Editor, NATHAN JACOBSON, Leet Oliver Memorial Hall, Yale University, New Haven 11, Conn.

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN